MATHEMATICAL LOGIC — ASSIGNMENT FOUR

(1) It is a known fact that if $\vdash_{\mathsf{PA}} A$ then $\vdash_{\mathsf{HA}} (A)^N$, in which $\vdash_{\mathsf{PA}} A$ denotes provability in Peano arithmetic, $\vdash_{\mathsf{HA}}$ denotes provability in Heyting arithmetic, and $(\_)^N$ is the Gödel-Gentzen translation.

Prove that Peano arithmetic is consistent if and only if Heyting arithmetic is consistent.

If Heyting arithmetic is not consistent, then $\vdash_{\mathsf{HA}} \bot$, thus $\vdash_{\mathsf{PA}} \bot$ since every intuitionistic derivation is also a classical proof and the axioms of Heyting and Peano arithmetic are the same.

Hence Peano arithmetic is not consistent.

Conversely, if Peano arithmetic is not consistent, then $\vdash_{\mathsf{PA}} \bot$, thus, using the known fact, $\vdash_{\mathsf{HA}} (\bot)^N$, i.e., $\vdash_{\mathsf{HA}} \bot$, that means Heyting arithmetic is not consistent.

(2) Prove the fixed point lemma:

Let Ξ be a theory in which every partial recursive function is representable and let A be a formula such that $\mathsf{FV}(A) = \{ y \}$.
Then there is a formula δ_A such that $\mathsf{FV}(\delta_A) = \emptyset$ and $\vdash_\Xi \delta_A = A[\ulcorner \delta_A \urcorner / y]$.

This is Lemma 31.6 in the slides.

(3) Show that $\neg A \vee \neg\neg A$, which is an instance of the Law of Excluded Middle thus valid in classical logic, is not provable in intuitionistic logic.

(Hint: Remember that the usual topology on $\mathbb{R}$ is a Heyting algebra.)

Consider the usual topology on $\mathbb{R}$ and interpret $[\![A]\!] = (0,1)$.

Since $[\![x \supset y]\!]$ is the maximal element c such that $[\![x]\!] \wedge c \leq [\![y]\!]$, then $[\![x \supset y]\!] = \mathsf{int}((\mathbb{R} \setminus [\![x]\!]) \cup [\![y]\!])$ with int the interior of its argument.

Hence

$$\begin{aligned} & [\![\neg A]\!] \\ = & [\![A \supset \bot]\!] \\ = & \mathsf{int}\left((\mathbb{R} \setminus (0,1)) \cup \emptyset\right) \\ = & \mathsf{int}\left((-\infty, 0] \cup [1, +\infty)\right) \\ = & (-\infty, 0) \cup (1, +\infty) \end{aligned}$$

Also

$$\begin{aligned} & [\![\neg\neg A]\!] \\ = & \mathsf{int}\left((\mathbb{R} \setminus [\![\neg A]\!]) \cup \emptyset\right) \\ = & \mathsf{int}\left([0,1]\right) \\ = & (0,1) \end{aligned}$$

Date: June 3rd, 2026.

Thus

$$
\begin{aligned}
&[\![\neg A \vee \neg\neg A]\!] \\
&= [\![\neg A]\!] \cup [\![\neg\neg A]\!] \\
&= (-\infty, 0) \cup (0, 1) \cup (1, +\infty) \\
&= \mathbb{R} \setminus \{0, 1\}
\end{aligned}
$$

Therefore $[\![\neg A \vee \neg\neg A]\!] \neq \mathbb{R} = [\![\top]\!]$, that is, $[\![\neg A \vee \neg\neg A]\!]$ is not valid.