The Evolution of Space

From Euclidean Geometry to Grothendieck's Topoi

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The Evolution of Space: Lecture 1

Syllabus:

- The absolute canvas: From Euclid to Descartes

A shift in perspective

From timeline to theme

- In the first part of this course, we travelled mostly **chronologically** through the history of mathematics
- We watched ideas, tools, notations, and proofs evolve from antiquity to the 20th century
- In this monographic section, we will travel **thematically**

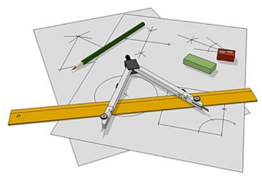
Our focus: What is "Space"?

- Intuition tells us space is the empty, rigid room in which the observable universe happens
- But mathematically, the concept of "space" has undergone a radical, mind-bending evolution
- We will trace this single idea: how space went from a physical absolute to a fluid, infinite-dimensional, and purely categorical abstraction

The Greek spatial paradigm

Euclid's unmeasurable void:

- For the ancient Greeks, geometry was the study of **idealised physical objects** (lines, circles, solids)
- **Crucial limitation**: Greek space was **not quantified**. There were no coordinates, no numbers attached to points



Geometry by construction, not calculation

- Magnitudes could be compared (line A is longer than line B), but they were conceptually distinct from discrete numbers (arithmetic)

Space as a passive canvas: In classical geometry, space has no properties of its own. It is merely the flat, absolute background where geometric constructions are drawn using compass and straightedge

The architecture of The Elements

The quest for absolute certainty (~300 BCE):

- Euclid's *Elements* is not just a geometry textbook; it is the first grand attempt to capture the structure of space using pure logic
- He built his universe starting from nothing but:
  - **23 Definitions** (e.g., "A point is that which has no part")
  - **5 Common Notions** (general logical axioms, e.g., "Things equal to the same thing are equal to each other")
  - **5 Postulates** (geometric axioms)

The architecture of The Elements

The synthetic method:

- Every subsequent theorem and lemma must be constructed strictly from these initial assumptions
- For over two millennia, this axiomatic method was considered the absolute pinnacle of human reasoning and the unshakeable truth of physical space

Euclidean geometry is **synthetic**: its truths are obtained as consequences of the axioms, not from an underlying well-defined world.
Compare against analytical geometry whose results are a consequence of the space being a set of vectors

The hidden flaws of the master

What Euclid missed:

- Despite its brilliance, modern mathematicians (notably **David Hilbert** in 1899) realised *The Elements* was full of logical gaps
- Euclid implicitly relied on spatial intuition and **diagrams** for things he never formally axiomatised

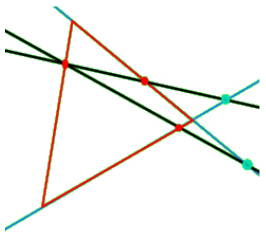
Implicit assumptions:

- **Betweenness**: If A, B, C are on a line, what does it formally mean for B to be "between" A and C?
- **Continuity**: If a line segment connects a point inside a circle to a point outside, it *must* intersect the circle. Euclid assumed this visually, but never provided an axiom to guarantee the space has "no holes"

Filling the gaps: Pasch's axiom

The danger of intuition:

- Euclid's reliance on diagrams masked missing logical steps. The most famous fix was provided by **Moritz Pasch** in 1882
- **Pasch's Axiom**: Let A, B, C be three non-collinear points, and let L be a line not passing through any of them. If L intersects the segment AB, then it must also intersect either AC or BC



A line entering a triangle must exit it

Why it matters:

- It seems incredibly obvious to the eye. But *logically*, without an axiom of **betweenness** or plane separation, the formal axiomatic machine cannot 'know' that the line does not simply slip through a 'hole' in the plane
- This highlights the leap from classical geometry (visual) to modern formalism (strictly syntactic)

The anomaly: The fifth postulate

The first four postulates are beautifully simple and constructive:

1. To draw a straight line from any point to any point
2. To produce a finite straight line continuously in a straight line
3. To describe a circle with any centre and distance
4. That all right angles are equal to one another

Then comes the fifth (the parallel postulate):

> *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles*

A flaw or a stroke of genius?

The scandal of geometry:

- The fifth postulate is wordy, complex, and looks far more like a theorem than a self-evident truth
- For 2000 years, mathematicians viewed this as a flaw in Euclid's work. They believed the fifth postulate **must** be derivable from the first four

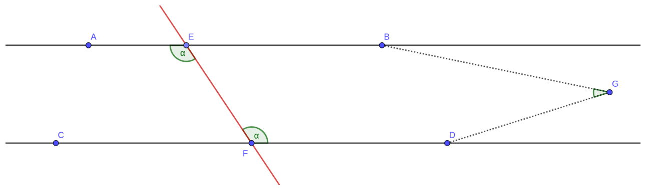
Euclid's profound insight:

- Euclid himself seemed suspicious of it. He **refused to use the fifth postulate** for the first 28 propositions of Book I
- The geometry built on just the first four postulates is today called **neutral geometry** (or absolute geometry)
- He only invoked the fifth when absolutely forced to (to prove properties of parallel lines and that the angles of a triangle sum to 180°)
- Far from a mistake, recognising that this property *could not be proven* and had to be assumed was perhaps Euclid's greatest stroke of genius

Neutral geometry in action: Proposition 27

Proving parallelism without the 5th postulate:

Proposition 27: If a straight line falling on two straight lines makes the alternate interior angles equal, then the straight lines will be parallel

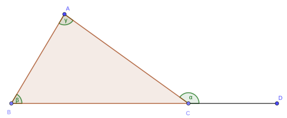


Proof (by contradiction)

- Let transversal EF cut lines AB and CD
  Assume alternate angles are equal ($\angle AEF = \angle EFD$)
- Suppose AB and CD are *not* parallel
  Then they meet at some point G, forming a triangle $\triangle EFG$

Neutral geometry in action: Proposition 27

- By Euclid's **Proposition 16** (the *Exterior Angle Theorem*), the exterior angle of a triangle is strictly greater than either opposite interior angle
  Thus, $\angle AEF > \angle EFD$
- But we assumed $\angle AEF = \angle EFD$!
  Contradiction
  Therefore, the lines cannot meet



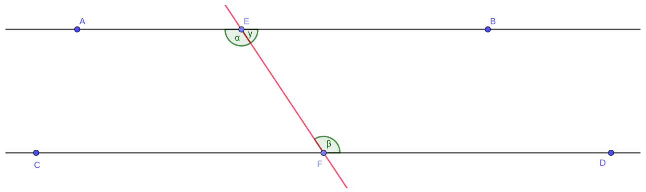
$\alpha > \beta$ and $\alpha > \gamma$, strictly

The power of absolute geometry: Euclid secured this fundamental property of parallel lines using only the first four postulates

The barrier broken: Proposition 29

The 5^{th} postulate enters the stage:

Proposition 29 is the converse of 27:
A straight line falling on parallel lines makes the alternate interior angles equal



Proof (by contradiction):

- Let $AB \parallel CD$ with transversal EF
  Suppose the alternate angles are *not* equal
  One must be greater, say $\angle AEF > \angle EFD$
- Add $\angle BEF$ to both sides: $\angle AEF + \angle BEF > \angle EFD + \angle BEF$

The barrier broken: Proposition 29

- The angles on the straight line AB equal two right angles (180°)
  So, two right angles $> \angle EFD + \angle BEF$
- **Enter postulate 5**: Since the two interior angles on the right side sum to *less* than two right angles, the lines AB and CD **must meet** on that side
- But AB and CD are given as parallel!
  Contradiction.
  Therefore, the angles must be equal

Proving necessity: The birth of models

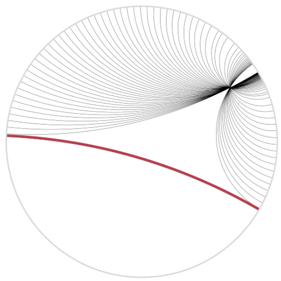
Is the 5^{th} postulate strictly necessary for Proposition 29?

- To prove that a postulate cannot be deduced from the others, modern mathematics uses **models** (a concept that would floor Euclid)
- We construct a space where postulates 1 through 4 are **true**, but postulate 5 is **false**
- In the **hyperbolic plane** (which we will explore soon), given a line L and a point P not on L, there are *infinitely many* lines through P that do not intersect L (they are all "parallel" to L)
- If we draw a transversal through these multiple parallel lines, the alternate interior angles *cannot* be equal for all of them
- Thus, Proposition 29 **fails** when postulate 5 is removed

The fifth postulate is **logically independent** and absolutely necessary for the rigid Euclidean grid

Proving necessity: The birth of models

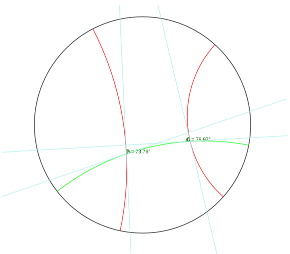
The **Poincaré disk** is a model of hyperbolic geometry



Poincaré's disk

- The points in the interior of the disk are the points in the geometry
- The lines are arcs of circles orthogonal to the boundary of the disk
- It is not too difficult to check that the first four postulates hold in this model
- However, the fifth does not: given a line, there are infinitely many parallel lines passing through an external point

Proving necessity: The birth of models



- For each pair of parallel lines (red in the figure), the transversal (green) forms alternate interior angles which are different
- Thus Proposition 29 is false
- As a consequence, one cannot **prove** the fifth postulate from the other four:
  - If postulates 1–4 prove postulate 5 then Proposition 29 would be true
  - Hence postulate 5 is independent from the first four by the *contrapositive* of the above statement
- And the fifth postulate is necessary to prove Proposition 29

The classical conception of space

Summary of the Euclidean paradigm:

- **Flat and rigid**: Space extends uniformly in all directions. A triangle drawn on the earth follows the exact same rules as a triangle drawn across the solar system
- **Continuous but not discrete**: Space is made of points and lines, but it is not parameterised by sets of numbers (there is no $\mathbb{R}^n$)
- **Absolute**: Space is the stage; it is independent of the objects within it. It acts upon objects (allowing them to have shape and distance), but objects cannot act upon space

This unshakeable, rigid void would dominate human thought until the 17th century, when Descartes and Fermat would commit the ultimate geometric heresy: they turned space into numbers

References

The main sources for this lecture are

- *Max Jammer*, Concepts of Space: The History of Theories of Space in Physics, third edition, Dover Publications (2003)
- *Jeremy Gray*, Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century, Springer (2010)
- *Marvin Jay Greenberg*, Euclidean and Non-Euclidean Geometries: Development and History, fourth edition, W.H. Freeman (2007)

For a fascinating philosophical companion, we highly recommend *Edwin A. Abbott*, Flatland: A Romance of Many Dimensions (1884), a brilliant novella that forces the reader to confront the limits of their own dimensional perception

The Evolution of Space: Lecture 2

UNIVERSITAS STUDIORUM INSUBRIAE

Syllabus:

- The arithmetisation of space: Descartes and Fermat

The great divide

Two separate universes:

- For the Greeks, mathematics was strictly divided into two distinct realms:
  - **Arithmetic**: The study of discrete multitudes (integers, counting)
  - **Geometry**: The study of continuous magnitudes (lengths, areas, volumes)

The wall of incommensurability:

- The discovery that the diagonal of a square is incommensurable with its side (e.g., $\sqrt{2}$) proved that continuous magnitudes could not be fully captured by the discrete ratios of integers (fractions)
- Consequently, geometry was elevated above arithmetic as the true, rigorous foundation of mathematics
- Numbers were numbers; lines were lines. They did not mix

The dimensionality trap

The Greek algebraic restriction:

- When the Greeks (like Euclid or geometric algebraists) multiplied two line segments, a and b, the result ab was an **area** (a rectangle)
- Multiplying three segments, abc, produced a **volume**
- Therefore, expressions like a^4 or $a^2 + b^3$ were considered **geometrically meaningless and absurd**
- You cannot add an area to a volume because they are not geometrically compatible, nor can you visualise four dimensions
- This strict adherence to physical dimensionality heavily restricted the development of abstract algebra

The 17th century catalyst

The linguistic prerequisite:

- Before space could be translated into algebra, algebra needed a language capable of describing space
- **François Viète** (1591) introduced symbolic algebra, using letters to represent both knowns (consonants) and unknowns (vowels)
- This abstraction was the critical spark. It shifted the focus from finding a specific numerical answer to understanding the *general relationship* between quantities

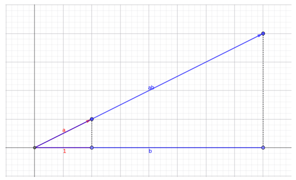
The dual revolution:

- In the 1630s, two Frenchmen—**Pierre de Fermat** and **René Descartes**—independently realised that Viète's new algebra could be unleashed upon Euclid's geometry

René Descartes and *La Géométrie*

Breaking the dimension barrier (1637):

- In his appendix *La Géométrie*, Descartes made a brilliant, foundational move: he redefined multiplication
- He introduced a "unit segment", 1
- To multiply segment a by segment b, he used similar triangles. The proportion $1 : a = b : ab$ ensures that the product ab is **another line segment**, not an area!



Descartes' multiplication: ab is a line

The algebraic liberation:

- Suddenly, a^2, a^3, and a^n were all just line segments
- Expressions like $a^3 + b^2$ were perfectly valid
  Algebra was freed from the spatial constraints of 3D geometry

The hidden power of similarity

A new look at an old concept:

- Greek geometry deeply explored **similar figures** and proportions (rigorously formulated by Eudoxus in Book V of Euclid's *Elements*)
- Similarity between 2D or 3D shapes is structurally rich. However, **similarity between two line segments** seems trivial: any two segments are trivially similar to one another!
- Descartes' brilliance was recognising that this "trivial" 1D similarity is actually the geometric engine needed for algebra

From proportions to vector spaces

The geometric meaning of multiplication:

- By setting up the proportion $1 : a = b : ab$ using similar triangles, Descartes did more than just find a length
- He defined an operation where a "scalar" magnitude (a, measured against the unit 1) stretches or compresses a geometric "vector" (b)
- **The modern echo**: This intuition is the exact historical seed for the modern definition of a **vector space** over a field. Specifically, it embodies the axioms of **scalar multiplication** ($\mathbb{R} \times V \to V$), bridging the algebraic field of scalars with the geometric space of vectors

$$a(b\vec{v}) = (ab)\vec{v}$$
$$1\vec{v} = \vec{v}$$
$$a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$$
$$(a+b)\vec{v} = a\vec{v} + b\vec{v}$$

Pierre de Fermat and the locus

Ad locos planos et solidos isagoge (1636):

- While Descartes used algebra to solve geometric construction problems, Fermat was interested in the **locus** of points
- Fermat's fundamental principle:

  > *"Whenever two unknown quantities are found in final equation, we have a locus, the extremity of one of these describing a line, straight or curved"*

- Let x and y be variable line segments originating from a reference axis. An equation like $ax + by = c$ inherently defines a straight line, while $x^2 + y^2 = r^2$ defines a circle

The isomorphism of space

The birth of the coordinate system:

- Neither Fermat nor Descartes explicitly drew two orthogonal axes with negative numbers (that modern "Cartesian plane" evolved later)
- However, they established the conceptual **isomorphism between geometry and algebra**
- **A point** is identified with a tuple of numbers: $P \leftrightarrow (x, y)$
- **A curve** is identified with an algebraic equation: $C \leftrightarrow f(x, y) = 0$

The arithmetisation of space:
Space is no longer an absolute, unmeasurable void
It is a dense, quantified grid
To study the geometry of a curve, one studies the algebra of its equation

The consequences of coordinatisation

1. The invention of new curves:

- Greek geometry was limited to curves that could be physically constructed or generated by kinematic motion (conics, cissoids, spirals)
- Analytic geometry allowed mathematicians to *invent* curves simply by writing down polynomials of higher degree (e.g., $y = x^3 - 2x + 1$)
- The degree of the polynomial immediately revealed geometric properties (degree 1 = line, degree 2 = conics)

2. Geometry becomes mechanical:

- Visual intuition was replaced by systematic, algebraic calculation
- To find where two curves intersect, do not draw them; solve their equations simultaneously

The freedom of the grid

Beyond orthogonal axes:

- A common misconception is that Descartes invented the rigid grid of perpendicular x and y axes
- In reality, Descartes rarely used orthogonal axes. He preferred **oblique (non-orthogonal) axes**, choosing whatever reference lines made the geometry easiest to express
- In solving the ancient **Pappus's locus problem** (finding a point based on its distances to 3 or 4 given lines), Descartes simply chose two of the given lines to be his axes
- The angle ω between the axes was arbitrary. The coordinates (x, y) were lengths measured parallel to these intrinsic geometric lines, not an external 90° grid

Example 1: The ellipse and conjugate axes

Letting the geometry dictate the algebra:

- Consider an ellipse. In a standard orthogonal grid aligned with its major and minor axes, its equation is neatly $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- What if the ellipse is tilted? The equation becomes messy, gaining a cross-term (like xy)
- However, Apollonius knew about **conjugate diameters** (a chord parallel to one diameter is bisected by the conjugate diameter).
- If we set *any* two conjugate diameters as our **oblique axes**, the equation of the ellipse remains the beautifully simple $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$

The lesson: The simplest algebraic form is achieved not by forcing the curve into a rigid orthogonal box, but by adapting the axes to the intrinsic symmetries of the curve

From oblique axes to change of basis

A profound algebraic insight:

- By allowing axes to shift and skew, Descartes implicitly recognised that the structure of space could be explored via **coordinate transformations**
- **Modern consequence**: This idea became the cornerstone of linear algebra and differential geometry
- To simplify the algebraic expression of a mathematical object, **we transform the space**
- Diagonalising a quadratic form is the modern, rigorous culmination of Descartes' flexible grid—finding the "perfect" axes to reveal the true geometry of an equation

Example 2: Diagonalising a quadratic form

Unmasking the geometry:

- Consider the equation: $5x^2 + 4xy + 5y^2 = 1$
- The $4xy$ cross-term indicates the shape is "tilted" relative to our current (x, y) axes. We rewrite this using a symmetric matrix:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

- We find the eigenvalues of the matrix: $\lambda_1 = 7, \lambda_2 = 3$
- By changing our basis to the eigenvectors (effectively rotating our grid by 45°), we define new coordinates (X, Y) where the equation becomes:

$$7X^2 + 3Y^2 = 1$$

The power of transformation: A complex, tilted object is instantly revealed as a standard ellipse. The algebra simplifies when the grid adapts to the space

The philosophical horizon: n-dimensions

The seed of higher-dimensional space:

- If a plane is just the set of pairs (x, y), and 3D space is the set of triples (x, y, z), what stops us from considering 4-tuples (x, y, z, w)?
- **Geometrically**, we cannot visualise 4D space
- **Algebraically**, adding a variable changes absolutely nothing about the underlying rules of polynomials or linear equations
- By reducing space to arithmetic ($\mathbb{R}^n$), Descartes and Fermat inadvertently laid the philosophical groundwork for geometries of n-dimensions, which would blossom in the 19^{th} century

Summary: The algebraic canvas

The paradigm shift:

- Euclid's space was an external, rigid reality that hosted geometric figures
- Analytic geometry transformed space into a **mathematical structure** ($\mathbb{R}^2$,$\mathbb{R}^3$) completely defined by real numbers
- The properties of space are now dictated by the properties of the field of real numbers

We have successfully quantified the void. The flat Euclidean canvas has been overlaid with an algebraic grid. But the next profound question approaches: What if the grid itself is not flat? What if the geometry we just quantified is not the only one possible?

References

The main sources for this lecture are

- *Carl B. Boyer*, History of Analytic Geometry, Dover Publications (2005)
- *René Descartes*, Discourse on Method, Optics, Geometry, and Meteorology, available on the course website
- *Michael Sean Mahoney*, The Mathematical Career of Pierre de Fermat, 1601–1665, second edition, Princeton University Press (2018)

For a deep dive into how algebra fundamentally restructured geometric thought, the first few chapters of *John Stillwell*, Mathematics and Its History, third edition, Springer (2010), provide excellent context for advanced students

The Evolution of Space: Lecture 3

Syllabus:

- The parallels in crisis: From Saccheri to Gauss

The two-millennium obsession

The blemish on Euclid's masterpiece:

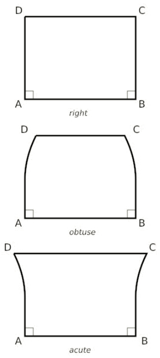
- As we saw in the previous lectures, the 5^{th} postulate lacks the self-evident simplicity of the first four
- From Ptolemy (2^{nd} century) to Proclus (5^{th} century) to Omar Khayyam (11^{th} century), mathematicians were convinced it was a **theorem** masquerading as an axiom
- The goal for 2000 years: Prove the 5^{th} postulate using only the first four (neutral geometry)
- Every single attempt either failed or implicitly assumed a statement *logically equivalent* to the 5^{th} postulate (e.g., "Playfair's axiom": through a point not on a line, there is exactly one parallel)

A change in strategy

Proof by contradiction:

- In the 18^{th} century, a new strategy emerged
- Instead of trying to prove the 5^{th} postulate directly, mathematicians assumed it was **false**
- The plan: Combine this false assumption with the first four postulates, deduce logical consequences, and eventually arrive at a **contradiction**
- This contradiction would prove that the denial of the 5^{th} postulate is absurd, thus proving the 5^{th} postulate itself must be true

Girolamo Saccheri (1667–1733)



The Saccheri quadrilateral

Euclides ab omni naevo vindicatus **(1733)** ("Euclid Freed of Every Flaw"):

- An Italian Jesuit priest and mathematician
- Saccheri constructed a figure to test the postulate: the **Saccheri quadrilateral**
- Base AB. Erect perpendicular segments AD and BC such that $AD = BC$
- Connect C and D. What are the angles at C and D?
- By symmetry (provable in neutral geometry), $\angle C = \angle D$. But are they 90°?

Saccheri's three hypotheses

Depending on the 5^{th} postulate, three cases arise for the summit angles $\angle C$ and $\angle D$:

1. **The hypothesis of the right angle** ($\angle C = \angle D = 90°$):
   This is equivalent to Euclidean geometry.
2. **The hypothesis of the obtuse angle** ($\angle C = \angle D > 90°$):
   Saccheri quickly derived a contradiction (it implies lines are finite in length, violating the 2^{nd} postulate)
   Today we know this models spherical geometry
3. **The hypothesis of the acute angle** ($\angle C = \angle D < 90°$):
   This is where the magic (and the tragedy) happened

Rejecting the obtuse case

Why was the obtuse hypothesis easily dismissed?

- Saccheri proved that the hypothesis of the obtuse angle implies the 5^{th} postulate as a theorem. This is not a contradiction yet
- It also implies that the sum of angles in a triangle is **greater than** 180°
- This directly contradicts Euclid's **Proposition 16** (the exterior angle theorem), which is a theorem of neutral geometry
- **The geometric culprit**: The obtuse case only works if we assume straight lines are **finite in length** (like great circles on a sphere)
- This violates Euclid's **2^{nd} postulate** (that a line can be extended continuously and indefinitely). Thus, in the context of Euclid's first four postulates, the obtuse case is logically impossible

The failure of nerve

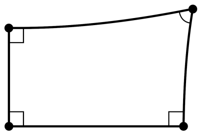
Exploring the acute hypothesis:

- Saccheri derived dozens of theorems under the acute hypothesis, searching for a contradiction
- For example, he proved that the sum of angles in a triangle would be strictly less than 180°
- Also, he proved that two parallel lines would either approach each other asymptotically or diverge to infinity
- **The tragedy**: He never found a logical contradiction!
  The theorems were strange, but mathematically consistent
- Eventually, driven by his goal to "vindicate" Euclid, he forced a weak, qualitative conclusion, claiming the asymptotic behaviour of lines was "repugnant to the nature of the straight line" and therefore declared the hypothesis false

Johann Lambert (1728–1777)

Pushing further into the unknown:

- Swiss polymath (also proved π is irrational)
- Studied the **Lambert quadrilateral**: three right angles, leaving the fourth angle to determine the geometry
- Like Saccheri, he explored the acute hypothesis without finding a logical contradiction



The Lambert quadrilateral

- **Lambert's profound observation**: Under the acute hypothesis, the area of a triangle is proportional to its **defect** (how much the sum of its angles fall short of 180°, i.e., $\pi - (\alpha + \beta + \gamma)$ in radians):

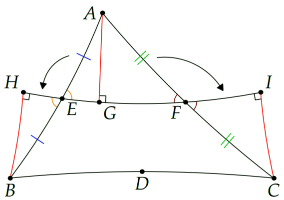
$$\text{Area} = k^2(\pi - (\alpha + \beta + \gamma))$$

where k is constant of curvature

Area of a triangle

Lambert's proof is beyond the scope of this course

- All Saccheri quadrilaterals with congruent bases and summit angles are congruent (congruent angles and sides)



- Given $\triangle ABC$, choose a side BC. Bisect the remaining sides at E, F and drop perpendiculars from A, B, C to EF. Then $HICB$ as in figure is a Saccheri quadrilateral with base HI
  - $\text{Area}(\triangle ABC) = \text{Area}(HICB)$
  - $\angle ABC + \angle BCA + \angle CAB = \angle HBC + \angle ICB$
- If $\triangle ABC$ and $\triangle PQR$ have the same angle-sum, they have the same area
- Hence the angle-defect of a hyperbolic triangle is an additive function of its area. We may conclude that $\text{Area} = k^2(\pi - (\alpha+\beta+\gamma))$

Lambert's imaginary sphere

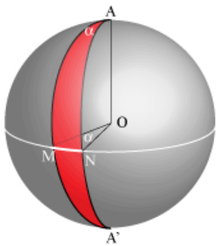
The geometric intuition:
Lambert knew that in spherical geometry (where the 5$^{\text{th}}$ postulate fails and triangle angles sum to $> 180°$), the area is proportional to the **excess**:

$$\begin{aligned}\text{Area} &= R^2((\alpha+\beta+\gamma)-\pi)\\ &= -R^2(\pi-(\alpha+\beta+\gamma))\end{aligned}$$

where R is the radius of the sphere

Lambert's imaginary sphere

The area A of a spherical wedge can be computed as



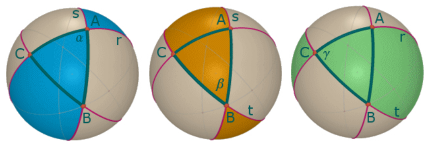
The area of a spherical wedge

- The area of the sphere of radius R is $4\pi R^2$
- The angle α at point A is the same as the angle between OM and ON
- Hence, calling A the area of the wedge

$$\frac{4\pi R^2}{A} = \frac{2\pi}{\alpha}$$

- So $A = 2\alpha R^2$

Lambert's imaginary sphere



To calculate the area A of $\triangle ABC$ with angles α, β, and γ,

- the area of the double wedge from A is $4\alpha R^2$
- the area of the double wedge from B is $4\beta R^2$
- the area of the double wedge from C is $4\gamma R^2$
- The three double-wedges cover the sphere, but they overlap at $\triangle ABC$ and its opposite, each being counted 3 times. Thus,

$$2\cdot\text{Area(wedges)} = \text{Area(sphere)} + 4\cdot\text{Area}(\triangle)$$

- Hence $4\alpha R^2 + 4\beta R^2 + 4\gamma R^2 = 4\pi R^2 + 4A$
- Thus $A = R^2((\alpha+\beta+\gamma)-\pi)$

Lambert's imaginary sphere

The geometric intuition:

- Hence

$$\text{Area} = R^2((\alpha+\beta+\gamma)-\pi) \\ = -R^2(\pi-(\alpha+\beta+\gamma))$$

  where R is the radius of the sphere
- Comparing this to his acute area formula, Lambert made an astonishing speculative leap:

  > *"I should almost conclude that the third hypothesis holds on some imaginary sphere"*

- The acute hypothesis was not nonsense; it was geometry on a surface with an **imaginary radius** ($R = ik$)!

The philosophical roadblock

Immanuel Kant (1724–1804):

- Why did no one before the 19th century accept that a non-Euclidean geometry could actually exist?
- The dominant philosophy of the era, solidified by Kant's *Critique of Pure Reason* (1781), held that Euclidean space was not an empirical observation, but an ***a priori* necessity** of the human mind
- We cannot even *conceive* of objects outside a Euclidean framework
- To suggest that Euclidean geometry was just one of several possible logical choices was considered philosophically absurd

Carl Friedrich Gauss (1777–1855)

The Prince of Mathematicians

The silent revolutionary:

- Gauss began working on the parallel postulate around 1792 (at age 15)
- Unlike Saccheri and Lambert, Gauss realised early on that the acute hypothesis did not lead to a contradiction
- He recognised it as a completely consistent, alternative geometric universe, which he first called "anti-Euclidean," then "astral," and finally **"non-Euclidean"** geometry

The clamour of the Boeotians

Why did Gauss never publish?:

- Gauss developed the foundations of non-Euclidean geometry in his private notes but published **nothing** on the subject during his lifetime
- His reasons are revealed in a letter to Friedrich Bessel (1829):

  > *"I shall probably not for a long time prepare my investigations on this subject for publication... for I fear the clamour of the Boeotians, if I were to speak my mind fully"*

- The "Boeotians" (a Greek tribe proverbially known for dullness) referred to the Kantian philosophers and traditional mathematicians who would vehemently attack him for destroying the supposed absolute truth of Euclidean space

Summary: The crack in the foundation

The paradigm shifts:

- The attempt to secure absolute truth through proof by contradiction failed, yielding consistent alternative geometries instead
- **Mathematical space decoupled from physical space**: Mathematics was no longer bound to describe the singular physical universe as perceived by human intuition
- Gauss's secret insight set the stage. The flat earth had been shattered in private, but it would take two young, bold mathematicians from the fringes of Europe to publicly announce the discovery

References

The main sources for this lecture are

- *Roberto Bonola*, Non-Euclidean Geometry: A Critical and Historical Study of its Development, Dover Publications (1911)
- *Marvin Jay Greenberg*, Euclidean and Non-Euclidean Geometries: Development and History, fourth edition, W.H. Freeman (2007)
- *Jeremy Gray*, Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century, Springer (2010)

For a direct look at the original texts, *John Fauvel and Jeremy Gray*, The History of Mathematics: A Reader, Red Globe Press (1987) contains translated excerpts from Saccheri, Lambert, and Gauss's private letters

The Evolution of Space: Lecture 4

Syllabus:

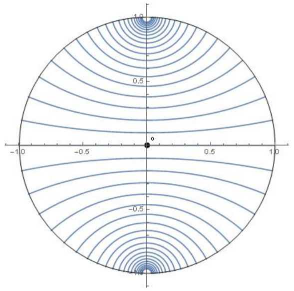
- The hyperbolic universe: Bolyai and Lobachevsky

The courage of the heretics

Out of nothing, a new universe:

- While Gauss kept his discoveries secret for 30 years, two young mathematicians independently published the first complete systems of **non-Euclidean geometry**
- **Nikolai Lobachevsky** (1829): A Russian mathematician who called his system **imaginary geometry**
- **János Bolyai** (1832): A Hungarian officer who wrote: *"Out of nothing I have created a strange new universe"*
- Their work was initially met with silence or ridicule, as it directly attacked the Kantian absolute truth of Euclidean space

The hyperbolic parallel postulate



Formalising the negation:

- The defining characteristic of hyperbolic geometry is the replacement of Euclid's 5^{th} postulate with its negation:
- **Lobachevskian Axiom**: Through a point P not on a line L, there exist **at least two** distinct lines through P that do not intersect L
- In fact, this implies there are **infinitely many** such lines, bounded by two "limiting parallels" (m_1 and m_2) which asymptotically approach L

The shattering of similarity

In Euclidean geometry, shapes can be scaled:

- We take **similar triangles** (same angles, different sizes) for granted. This requires the 5^{th} postulate

In hyperbolic geometry, similarity is congruence:

- If two triangles have the same three angles, they **must** have the same area and side lengths
- There are no similar triangles of different sizes
- **Profound consequence**: Space has an **absolute scale**. There exists a natural unit of length derived from the curvature of space

Area and the defect

Quantifying the "bend":

- Recall Lambert's discovery:
  the area of a triangle $\triangle$ is proportional to its **angle defect**
- Let α, β, γ be the interior angles in radians:

$$\text{Area}(\triangle) = k^2(\pi - (\alpha + \beta + \gamma))$$

- The sum of angles is always strictly less than π (180°)
- The maximum area of any triangle is $k^2\pi$, reached by an **ideal triangle** with all three vertices at "infinity" (angles $= 0$)

The constant $-\frac{1}{k^2}$ is called **curvature** and measures the *bending* of the space

In hyperbolic and spherical geometry, $\text{area}(\triangle) = \frac{\text{angle excess}}{\text{curvature}}$

The problem of consistency

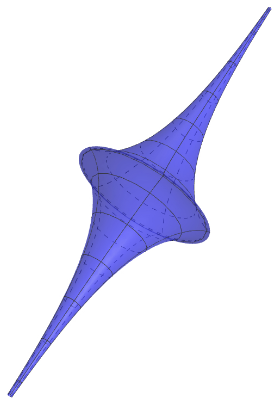
Logical ghosts vs physical truth:

- For most of the 19^{th} century, hyperbolic geometry was seen as a logical curiosity, not a "real" geometry
- Sceptics asked: How do we know a contradiction won't appear after proving 10,000 more theorems?
- Acceptance required a proof of **relative consistency**:
  If Euclidean geometry is consistent, then hyperbolic geometry is consistent
- This was achieved by constructing **models**:
  interpreting "points" and "lines" using Euclidean objects

The first model: Beltrami's pseudosphere

Eugenio Beltrami (1868):

- Proved that the geometry of a surface with **constant negative curvature** locally satisfies hyperbolic axioms
- The **pseudosphere** is generated by revolving a tractrix about its asymptote
- **Limitation**: Like a map of the world on a flat sheet, the pseudosphere cannot model the *entire* infinite hyperbolic plane without singular edges



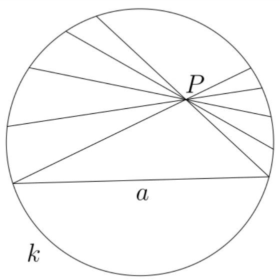
The pseudosphere

The limitation of the pseudosphere

A local, not global, universe:

- Beltrami's pseudosphere was a monumental breakthrough, but it only proves hyperbolic geometry is consistent **locally**
- Notice the *equator* (the widest circular base) of the pseudosphere: it forms a sharp, singular edge (a cusp)
- If a hyperbolic "straight line" (geodesic) hits this edge, it cannot be extended further. This directly violates Euclid's **2nd postulate** (lines can be produced indefinitely)
- **Hilbert's Theorem (1901)**: David Hilbert later proved that it is strictly impossible to smoothly embed the *entire* infinite hyperbolic plane into standard 3D Euclidean space without introducing singularities

The Klein model (projective)



Hyperbolic lines as Euclidean chords

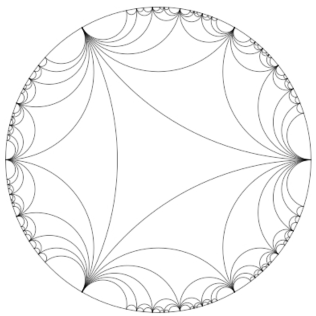
Felix Klein (1871):

- Developed a model using the interior of a Euclidean circle
- **Points**: Interior points
- **Lines**: Straight Euclidean **chords**
- **Parallelism**: Two chords that do not intersect inside the circle are parallel
- This model shows that hyperbolic geometry is a part of **projective geometry** (thus preserving lines) but distorts angles

The Poincaré disk model (conformal)

Henri Poincaré (1882):

- **Points**: Interior of a disk
- **Lines**: Circular arcs **orthogonal** to the boundary
- **Conformal**: This model preserves **angles**, i.e., they are measured as in the Euclidean plane, but distorts Euclidean distances
- As points approach the boundary, they move an infinite hyperbolic distance
- This model (and the upper half-plane model) became the standard for modern complex analysis

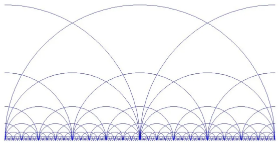


Poincaré's conformal model

The Poincaré half-plane model (conformal)

Henri Poincaré (1882):

- **Points**: $a + bi \in \mathbb{C}$ with $b > 0$
- **Lines**: Half circles with origin on the x-axis and straight Euclidean rays orthogonal to the x-axis
- **Conformal**: This model preserves **angles** but distorts Euclidean distances



Poincaré's half-plane model

The projective unification

Geometry via the cross-ratio:

- The models of Klein and Poincaré are not random optical tricks; they are deeply rooted in **projective geometry**
- In 1859, Arthur Cayley discovered how to define distance inside a projective space relative to a fixed boundary (the "Absolute") using the **cross-ratio** of four points
- Felix Klein realised that if the Absolute is a real circle, Cayley's projective metric yields exact **hyperbolic geometry**!

Isomorphic universes:

- The Klein and Poincaré models are mathematically equivalent mappings of the same underlying geometry
- One can translate between them via 3D projections: projecting the Klein disk vertically onto a hemisphere, and then stereographically projecting that hemisphere back down, perfectly generates the Poincaré disk

Summary: The autonomy of space

The death of absolute space:

- The existence of consistent models proved that Euclid's 5$^{\text{th}}$ postulate **cannot** be derived from the others
- It is a **choice**, not a logical necessity
- **The decoupling**: Mathematics no longer had to describe physical space. It became the study of **possible structures**
- Space acquires an intrinsic structure, its **curvature**
- This conceptual freedom paved the way for Riemann to generalise space into n-dimensional curved manifolds

By shattering the monopoly of Euclid, Bolyai and Lobachevsky didn't just find a new geometry; they redefined the purpose of mathematics itself

References

The main sources for this lecture are

- *Nikolai Lobachevsky*, Geometrical Investigations on the Theory of Parallels (1840)
- *János Bolyai*, The Absolute Science of Space (1832)
- *Roberto Bonola*, Non-Euclidean Geometry: A Critical and Historical Study of its Development, Dover Publications (1911)
- *Marvin Jay Greenberg*, Euclidean and Non-Euclidean Geometries: Development and History, fourth edition, W.H. Freeman (2007)

Bolyai's work was originally an appendix to his father Farkas's book *Tentamen*, while Lobachevsky published in the *Kazan Messenger*. Both works are monumental in the shift toward formal axiomatic freedom

The Evolution of Space: Lecture 5

Syllabus:

- The curvature of the manifold:
  Riemann and intrinsic geometry

The geometric perspective

Extrinsic vs intrinsic geometry:

- Until the 19th century, geometry was **extrinsic**
  Curves and surfaces were objects sitting *inside* an absolute, ambient 3D Euclidean space
- To measure how a surface bent, mathematicians (like Euler) looked at how it curved into the surrounding 3D void
- But what if the ambient space does not exist?
  What if we are trapped *inside* the surface?
- **Intrinsic geometry** is the study of properties that can be measured by a hypothetical 2D inhabitant who cannot see or access the 3rd dimension

Gauss's stunning discovery

Theorema Egregium (1827):

- Gauss published *Disquisitiones generales circa superficies curvas* (General investigations of curved surfaces)
- He defined Gaussian curvature (K) using the principal curvatures ($k_1 \cdot k_2$), an explicitly **extrinsic** definition
- Then, he proved his "remarkable theorem": K can be determined entirely by measuring distances and angles **on the surface itself**

Gauss's stunning discovery

Defining curvature:

- For a surface in 3D, at any point there are two **principal curvatures**, κ_1 and κ_2 (measuring the maximum and minimum bending of the surface)
- Gauss defined the **Gaussian curvature** as their product: $K = \kappa_1 \kappa_2$
- To calculate κ_1 and κ_2, you must know how the surface sits in the ambient 3D space (extrinsic)

The *Theorema Egregium*:

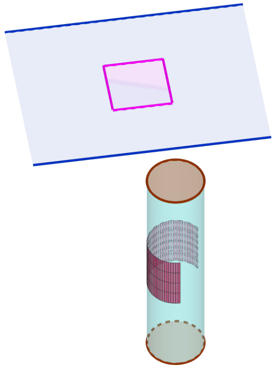
- Gauss proved that K can be completely rewritten using **only** the coefficients of the metric (the first fundamental form $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$) and their derivatives
- Because ds^2 only measures distances *along the surface*, K depends purely on internal measurements. It is an **intrinsic property**

Gauss's stunning discovery

The philosophical consequence:

- You can bend a flat Euclidean sheet of paper into a cylinder
- In 3D space, it looks curved
- But intrinsically, its geometry has not changed ($K = 0$)

Curvature is an inherent property of the space, independent of how it is embedded



Bending without stretching

Bernhard Riemann (1826–1866)

Bernhard Riemann

The 1854 Habilitation lecture:

- To qualify as a lecturer at Göttingen, Riemann had to present a lecture to the faculty
- He submitted three topics. Gauss, violating tradition, bypassed the first two and selected the third: **"On the hypotheses which lie at the foundations of geometry"**
- Gauss wanted to see what his brilliant student could do with the problem of space

Riemann delivered a lecture mostly devoid of equations, but it contained the most radical rethinking of space since Euclid
Gauss was reportedly "astounded" and commented, for the first time in his life, "completely satisfying"

The concept of the manifold

Breaking the dimensional barrier:

- Riemann stripped space of its physical baggage
  He defined a **manifold** (*Mannigfaltigkeit*, or "extended quantity")
- A manifold is simply a continuous space where each point can be uniquely identified by an n-tuple of numbers $(x_1, x_2, \ldots, x_n)$
- It does **not** need an $(n+1)$-dimensional space to sit inside

The paradigm shift: Space is no longer a void containing points; space *is* the set of points, structured by a local coordinate system

The metric tensor

How do we measure distance?:

- In Euclidean space, Pythagoras gives us the infinitesimal distance ds:

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2$$

- Riemann generalised this. He proposed that the metric geometry of *any* manifold is defined by a quadratic form:

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}\, dx_i\, dx_j$$

- The coefficients g_{ij} form the **metric tensor**
  They are functions, meaning they can **vary from point to point**

The metric tensor and curvature

From K to the Riemann tensor:

- In 2D, a single number (K) completely describes intrinsic curvature
- In n-dimensions, curvature is far more complex. Riemann introduced a multi-dimensional array of numbers derived entirely from the metric g_{ij} and its derivatives
- This is now known as the **Riemann curvature tensor**, denoted R^i_{jkl}

The geometric meaning:

- The Riemann tensor measures how much a vector changes when it is **parallel transported** around a tiny closed loop in the manifold
- If the space is flat (like $\mathbb{R}^n$), the vector returns unchanged, and $R^i_{jkl} = 0$
- If the space is curved, the vector returns rotated
  The tensor exactly quantifies this intrinsic rotation

An active, flexible fabric

Space acquires properties:

- The g_{ij} coefficients act as a local "ruler" that shrinks, expands, or skews depending on where you are in the manifold
- From these coefficients, Riemann showed how to calculate the intrinsic curvature of the n-dimensional space
  (using what we now call the **Riemann curvature tensor**)
- If the curvature is zero everywhere, the space is locally Euclidean
  If not, the space is fundamentally curved

The death of the passive canvas:
Space is now an active mathematical object
The geometry of the space is entirely dictated by the functions g_{ij}

Unifying the geometries

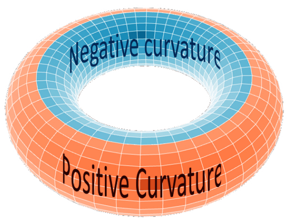
One framework to rule them all:

- Riemann's metric framework unified all previous geometric discoveries:
  - **Positive constant curvature**:
    Spherical geometry (finite, no parallels)
  - **Zero curvature**:
    Euclidean geometry
  - **Negative constant curvature**:
    Hyperbolic geometry (Lobachevsky and Bolyai)
- Furthermore, Riemann's space did not have to have *constant* curvature
  It could be lumpy, with curvature varying wildly from point to point

Variable curvature: The torus

A familiar, lumpy universe:

- Consider the surface of a torus
- Unlike a sphere or a Euclidean plane, its curvature is **not constant**
- **Outside (equator)**: Positive curvature ($K > 0$), locally like a sphere
- **Inside (near hole)**: Negative curvature ($K < 0$), locally like a saddle
- **Top/bottom circles**: Zero curvature ($K = 0$), locally flat



Curvature zones of a torus

Riemann's framework seamlessly handles this. By allowing the metric g_{ij} to be smooth functions of the coordinates, the math perfectly tracks the changing intrinsic curvature of the space

The prophetic conclusion

Physics meets geometry:

- At the end of his lecture, Riemann made a staggering philosophical leap
- Since physical space is an empirical entity, its geometry must be determined by empirical measurement, not axiomatic assumptions
- He suggested that the true metric structure of physical space (the g_{ij}) might be determined by the **binding forces of matter** within it
- 60 years later, **Albert Einstein** would realise exactly this:
  General relativity is precisely Riemann's geometry, where the metric tensor g_{ij} is shaped by gravity

Summary: The structural revolution

Space has structure:

- The space in which geometry is developed is no longer a void framework passively hosting objects
- Its **curvature** determines its structure, and consequently, the properties of the figures within it
- Identifying the inherent structure of space reveals that a vast variety of different spaces exist, massively enriching the mathematical world
- These spaces can be rigorously modelled by **manifolds** with variable curvature, governed entirely by the metric tensor

References

The main sources for this lecture are

- *Bernhard Riemann*, On the Hypotheses which lie at the Foundations of Geometry (1854 habilitation lecture)
- *Michael Spivak*, A Comprehensive Introduction to Differential Geometry, Vol. II, Publish or Perish (1999)
- *Jeremy Gray*, Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century, Springer (2010)
- *Detlef Laugwitz*, Bernhard Riemann 1826–1866: Turning Points in the Conception of Mathematics, Birkhäuser (1999)

For a highly accessible yet mathematically rigorous discussion of the shift from extrinsic to intrinsic curvature, we recommend Spivak's second volume, which meticulously traces Gauss and Riemann's original thoughts

The Evolution of Space: Lecture 6

Syllabus:

- The Erlangen program:
  Felix Klein and the symmetry lens

The geometry zoo

The crisis of abundance:

- By 1870, the mathematical world was overwhelmed by a sudden explosion of new geometries
- Mathematicians were studying Euclidean, hyperbolic, spherical, projective, affine, conformal, and inversive geometries
- **The problem**: These geometries seemed like isolated disciplines, each with its own axioms, definitions, and theorems
- Riemann had provided a unifying **differential** framework (the metric tensor), but mathematicians sought a unifying **algebraic** framework to understand the structural relationships between these spaces

Felix Klein (1849–1925)

Felix Klein

The 1872 inaugural address:

- At just 23 years old, Felix Klein was appointed as a professor at the University of Erlangen
- To mark his appointment, he published a prospectus that would fundamentally rewrite the definition of geometry
- This document is famously known as the **Erlangen program**
- Klein's revolutionary idea was to introduce the concept of **groups** (recently developed by Galois, Cauchy, and Sylow for algebra) into the heart of geometry

The core philosophy

A radical redefinition:

- Klein argued that geometry is not about measuring fixed shapes like triangles and circles
- Instead, geometry is the study of properties that **do not change** when the space is subjected to a specific set of transformations

The Erlangen definition:

> *"Given a manifold and a group of transformations of the manifold, to investigate the configurations belonging to the manifold with regard to those properties that are not altered by the transformations of the group"*

Geometry ≡ Invariants under a group action

Space as a pair (X, G)

The structural viewpoint:

- Mathematically, a space is no longer just a set of points X
- A space is defined by the pair (X, G), where G is a group acting on X
- **Group axioms required**:
  - *Associativity*: Doing one transformation then another is also a transformation in the group
  - *Identity*: Doing nothing is in the group
  - *Inverse*: Every transformation can be perfectly undone
- Changing the group G entirely changes the geometry, even if the underlying set of points X remains exactly the same

The hierarchy of geometries

Subgroups create structure:

- If H is a subgroup of G ($H \subset G$), then any property invariant under the large group G is **automatically** invariant under the smaller group H
- However, the smaller group H will have **additional invariants** (geometrical properties) that G destroys

The classical nesting:

$$\text{Euclidean group} \subset \text{Affine group} \subset \text{Projective group}$$

- The **smaller** the transformation group, the **more** rigid the geometry, and the more invariants you can measure (like distance and angle)
- The **larger** the group, the more fluid the geometry, leaving only deep topological or incidence properties intact

Projective geometry

The most fluid classical space:

- **The group**: Invertible linear transformations on homogeneous coordinates ($PGL(n,\mathbb{R})$)
- **Transformations**: Projections, cross-sections, perspectives
- **What is destroyed?**: Distance, angle, parallel lines (any two lines meet), and the ratio of lengths
- **What is invariant?**:
  - **Incidence**: Points on a line remain on a line
  - **Cross-ratio**: The fundamental invariant of four collinear points: $(A,B;C,D)$

Projective geometry

The cross-ratio:

- For four collinear points A,B,C,D, the cross-ratio is defined using signed distances, i.e., choosing a direction for the line:

$$(A,B;C,D) = \frac{AC \cdot BD}{AD \cdot BC}$$

- **Pappus's observation (~300 CE)**: In his *Collection*, Pappus proved that if four concurrent lines are cut by any transversal, the cross-ratio of the intersection points is constant
- **Projective invariance**: If we project a line onto another from a central point, absolute lengths and simple ratios change wildly
- However, the cross-ratio is **perfectly preserved**: $(A,B;C,D) = (A',B';C',D')$

Affine geometry

Recovering parallelism:

- **The group**: Linear transformations plus translations ($x \mapsto Ax + b$)
- We restrict the projective group by fixing the "line at infinity"
- **What is destroyed?**: Exact distances and angles (circles can be stretched into ellipses, squares into rhombuses)
- **What is invariant?**:
  - Incidence and cross-ratio (inherited from projective)
  - **Parallelism**: Parallel lines remain parallel
  - **Ratio of segments**: If M is the midpoint of AB, it remains the midpoint after transformation

Affine geometry

The general affine transformation:

- Maps a vector $\mathbf{x}$ to $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where A is any invertible $n \times n$ matrix, and $\mathbf{b}$ is a translation vector
- **Example (Shear mapping)**: Let $\mathbf{b} = 0$ and $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

  This transformation applied on a square pushes the top sideways, turning it into a parallelogram
- **The geometric effect**: Parallel lines remain parallel (the affine invariant) However, the 90° angles are destroyed, and the diagonal lengths change
- **The Euclidean difference**: Euclidean transformations *forbid* matrices that stretch or shear. They only allow matrices that act as rigid motions

Euclidean geometry

The rigid world of the Greeks:

- **The group**: Orthogonal transformations plus translations (rotations, reflections, translations)
- We restrict the affine group by requiring the matrix A to be orthogonal ($A^T A = I$)
- **What is invariant?**:
  - All projective and affine invariants
  - **Distance** between points
  - **Angle** between lines
  - **Area and volume**

To Klein, Euclid's Elements is simply the study of the invariants of the orthogonal group

Euclidean geometry

The Orthogonal Group $O(n)$:

- Algebraically, the orthogonal group $O(n)$ consists of all $n \times n$ real matrices A such that $A^T A = I$ (the identity matrix), which implies $A^{-1} = A^T$
- **Geometric meaning**: These matrices perfectly preserve the standard dot product of vectors: $(A\mathbf{u}) \cdot (A\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$

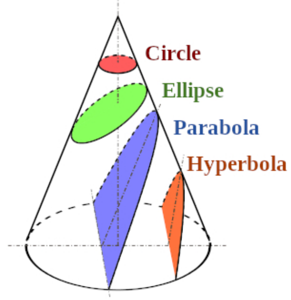
Consequences for space:

- Because the dot product dictates both lengths ($\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$) and angles ($\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$), preserving the dot product **exactly preserves lengths and angles**
- $O(n)$ contains both rotations and reflections. Its subgroup $SO(n)$ (where $\det(A) = 1$) represents the pure, rigid rotations of physical objects

Example: The eyes of the observer

What is a conic section?:

- **Euclidean view**: Ellipses, parabolas, and hyperbolas are totally distinct shapes, defined by specific distances (foci)
- **Affine view**: All ellipses are the same shape (just stretched circles). But ellipses, parabolas, and hyperbolas are still distinct because of their asymptotic behaviour
- **Projective view**: **They are the exact same curve**. A parabola is just an ellipse tangent to the line at infinity. A hyperbola is an ellipse intersecting the line at infinity



One curve, different infinite boundaries

Sophus Lie and continuous symmetries

The mathematical machinery:

- Klein developed the Erlangen Program in close collaboration with his friend **Sophus Lie**
- The transformation groups they were studying (rotations, projections) were not discrete (like the permutations of Galois); they were **continuous**
- Lie would go on to formalise these continuous transformation groups, giving birth to **Lie groups** and **Lie algebras**
- This beautifully unified algebra, geometry, and topology: a Lie group is simultaneously a group (algebra) and a smooth manifold (geometry)

Summary: Symmetry becomes primary

The legacy of the Erlangen Program:

- Space is defined by its **symmetries**
- Arthur Cayley famously declared: *"Projective geometry is all geometry"* because by making the group large enough, all other geometries appear as special sub-cases
- This algebraic approach shifted the mathematical focus toward invariants
- In the 20^{th} century, physicists and mathematicians working in mathematical physics (like Emmy Noether) would adopt Klein's exact philosophy: the fundamental laws of physics are simply the **invariants** of the universe's symmetry groups

References

The main sources for this lecture are

- *Felix Klein*, A Comparative Review of Recent Researches in Geometry (The Erlangen Program, 1872)
- *Isaak Yaglom*, Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century, Birkhäuser (1988)
- *John Stillwell*, Mathematics and Its History, third edition, Springer (2010)
- *Jeremy Gray*, Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century, Springer (2010)

For Master's students, Yaglom's book is highly recommended. It perfectly captures how the algebraic theory of groups, originally born from the study of polynomial equations, invaded and conquered geometry

The Evolution of Space: Lecture 7

Syllabus:

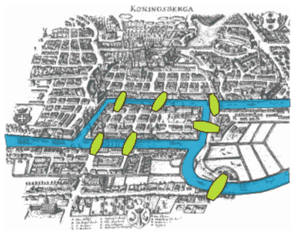
- The birth of topology:
  From Euler to Poincaré

The ultimate abstraction

Stripping away the rigid structures:

- In Euclidean and non-Euclidean geometries, space is measured by **distances and angles** (Riemann's metric)
- In projective and affine geometries, space is governed by **straight lines and cross-ratios** (Klein's algebraic groups)
- But what if we discard the metric *and* the straight lines?
- Imagine the space is made of infinitely stretchable, compressible rubber What geometric properties survive continuous deformation?
- This is the domain of **topology** (historically known as *geometria situs* or *analysis situs* — the geometry of position)

Leonhard Euler (1707–1783)



Geometry without distance

The bridges of Königsberg (1736):

- The city of Königsberg had 7 bridges connecting two islands and the mainland
- **The puzzle**: Can you walk through the city, crossing every bridge exactly once?
- Euler proved it was impossible. His genius was realising that exact distances, island shapes, and bridge lengths were **completely irrelevant**
- He reduced the space to a network of points (nodes) and lines (edges) This was the birth of **graph theory**, as already remarked, and the first purely topological reasoning

The first topological invariant

Euler's polyhedron formula (1752):

- Euler discovered a relationship between the Vertices (V), Edges (E), and Faces (F) of any convex polyhedron:

$$V - E + F = 2$$

- A cube ($8-12+6=2$), a tetrahedron ($4-6+4=2$), and a dodecahedron ($20-30+12=2$) all share this fundamental property
- The number 2 is not a property of the sharp edges; it is a profound global invariant of the sphere itself (the **Euler characteristic**, χ)

Euler's polyhedron formula

Proof:

- **Remove a face**: The remaining shape becomes a network of vertices and edges such that $V - E + F = 1$
- **Triangulate**: If any faces are not triangles, add diagonal edges inside them until all faces are triangles. Adding a diagonal edge adds 1 edge and 1 face, so $V - E + F = V - (E+1) + (F+1)$
- **Remove triangles**: Now, remove triangles one by one from the edge of the network. Either 1 edge and 1 face are removed, or 2 edges, 1 face and 1 vertex are removed. In both scenarios, $V - E + F$ stays constant
- **Final state**: When only one triangle remains, $V = 3$, $E = 3$, $F = 1$ Adding the initially removed face, $V - E + F = 2$

The 19th century shift

Naming the void:

- In 1847, Johann Listing (a student of Gauss) published *Vorstudien zur Topologie*, officially introducing the word **topology** (from Greek *topos*, place) in Mathematics
- Mathematicians began discovering shapes that defied standard 3D Euclidean intuition:
  - **The Möbius strip (1858)**: A surface with only one side and one edge
    It proved that **orientability** (having a distinct inside/outside or up/down) is a topological property, not a given
  - **The Klein bottle (1882)**: A closed, non-orientable surface that has no boundary but cannot be embedded in 3D space without intersecting itself

The global holes of Riemann

Complex analysis meets topology:

- As we saw in Lesson II-5, Riemann studied manifolds
- In his work on complex functions, Riemann realised that the behaviour of integrals depended entirely on the **global shape** of the domain
- He classified surfaces by their **genus** (g), the "number of holes"
- A sphere has genus 0, a torus has genus 1, a double-torus has genus 2
- Riemann showed that the Euler characteristic is directly tied to the genus:

$$\chi = 2 - 2g$$

Henri Poincaré (1854–1912)

The birth of modern topology:

- While Euler and Riemann studied 1D networks and 2D surfaces, Poincaré wanted to understand qualitative structures in 3, 4, and n dimensions
- His motivation came from celestial mechanics (the 3-body problem): equations were too complex to solve exactly, so he needed to study the **qualitative geometry** of their solution spaces
- In 1895, he published *Analysis Situs*, laying the rigorous foundation for what we now call **algebraic topology**

Henri Poincaré

The machinery of Analysis Situs

Formalising the deformation:

- Poincaré needed a way to mathematically tell if two n-dimensional spaces were topologically equivalent (**homeomorphic**)
- **Betti numbers**: He generalised Riemann's "holes" to higher dimensions
  He defined numbers that count the 1D loops, 2D voids, 3D cavities, etc., that cannot be shrunk to a point
- **The fundamental group**: He introduced the idea of taking a closed loop in a space and seeing if it can be continuously deformed into another loop (homotopy) in the same space
- By associating **algebraic groups** with spaces, Poincaré created a computable machine to distinguish non-equivalent spaces

Betti numbers

Informally, the k-th Betti number is the number of k-dimensional holes on a topological surface

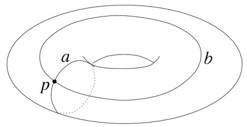
A k-dimensional hole is a k-dimensional cycle that is not a boundary of a $(k+1)$-dimensional object

The first few Betti numbers have the following definitions for 0-dimensional, 1-dimensional, and 2-dimensional simplicial complexes:

- b_0 is the number of connected components
- b_1 is the number of one-dimensional or "circular" holes
- b_2 is the number of two-dimensional cavities

For example, a torus has one connected component so $b_0 = 1$, two circular holes (one equatorial and one meridional) so $b_1 = 2$, and a single cavity enclosed within the surface so $b_2 = 1$

The fundamental group



Consider a torus, fix a point p on its surface, and a direction for loops:

- The fundamental closed paths from p to p are the *identity path* (staying at p), the longitudinal loop a, and the meridional loop b
- A "complex" loop is obtained composing simple loops
- Therefore a loop is uniquely associated with a pair of integers, counting how many times the path follows a and b and in which direction
- In short, $\langle \mathbb{Z} \times \mathbb{Z}; + \rangle$ is the fundamental group of the torus

Summary: The qualitative universe

Space as a continuous medium:

- Topology fundamentally separated the concepts of **proximity and continuity** from the concept of **distance**
- It provides the deepest, most foundational layer of spatial structure
  Before you can measure a space (metric) or transform it (groups), you must know its topological shape
- *A coffee mug and a doughnut are topologically identical*
  The metric geometry is entirely different, but the fundamental spatial connectivity is the exact same

With Poincaré, space is no longer just measured or transformed; it is classified by its algebraic holes
This sets the stage for our next step into homology and homotopy

References

The main sources for this lecture are

- *I.M. James* (ed.), History of Topology, Elsevier (1999)
- *John Stillwell*, Mathematics and Its History, third edition, Springer (2010)
- *Jean Dieudonné*, A History of Algebraic and Differential Topology, 1900-1960, Birkhäuser (1989)

For advanced students, Poincaré's original 1895 paper *Analysis Situs* (translated by John Stillwell) is a masterclass in reading mathematical intuition before the era of strict set-theoretic formalism

The Evolution of Space: Lecture 8

Syllabus:

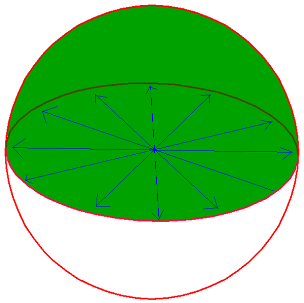
- Algebraic topology:
  Homology, homotopy, and the Poincaré conjecture

The limit of Betti numbers

When counting holes is not enough:

- In the early 1900s, mathematicians following Poincaré were using Betti numbers $(b_0, b_1, b_2, \dots)$ to classify spaces
- However, numbers have a fatal flaw: they only measure **free** dimensions (like the $\mathbb{Z} \times \mathbb{Z}$ of the torus)
- They completely miss spaces that have **torsion**, that is, topological "twists" where a path loops around several times before being able to shrink to a boundary
- For example, the real projective plane ($\mathbb{RP}^2$) has the exact same Betti numbers as a single point. To Betti numbers, the twist of $\mathbb{RP}^2$ is invisible

The real projective plane



Antipodal identification

Visualising the twist:

- $\mathbb{RP}^2$ can be modelled as a sphere where every pair of antipodal (opposite) points is "glued" together
- Equivalently, take a **hemisphere** and identify opposite points on its circular boundary (the equator)
- **The topological path**: Imagine an ant walking across the hemisphere When it hits the equator at point P, it instantly reappears at the exact opposite point $-P$
- It must cross the hemisphere **twice** to return to its original starting position and orientation

Emmy Noether's paradigm shift (1925)

From numbers to groups:

- During a series of lectures by Heinz Hopf in 1925, the great algebraist **Emmy Noether** made a profound observation
- She argued that Betti numbers and torsion coefficients should not be viewed as isolated integers
- Instead, they are simply the **rank** and the **torsion components** of a finitely generated Abelian group
- **The new rule**: Do not assign a space a sequence of numbers; assign it a sequence of **homology groups** $(H_0, H_1, H_2, \dots)$

Emmy Noether

The machinery of homology

Building the algebraic machine:

- To compute these groups, spaces are triangulated into **simplices** (points, line segments, triangles, tetrahedra)
- A formal linear combination of n-dimensional simplices is called an **n-chain**, forming a vast vector space (or free Abelian group) denoted C_n
- The magical operator is the **boundary operator** ($\partial_n : C_n \to C_{n-1}$), which maps a simplex to its bounding faces
- Example: The boundary of a filled triangle is its three edges

Cycles, boundaries, and quotient spaces

The fundamental equation of topology:

- The boundary of a boundary is always empty
  Algebraically: $\partial_{n-1} \circ \partial_n = 0$

Defining the homology group H_n:

- **Cycles (Z_n)**: Chains with no boundary ($\ker \partial_n$)
  These form closed loops or shells
- **Boundaries (B_n)**: Chains that are the boundary of a higher-dimensional object (im ∂_{n+1})
- Because $\partial_{n-1} \circ \partial_n = 0$, every boundary is a cycle ($B_n \subseteq Z_n$), but *not every cycle is a boundary* (e.g., a loop around the hole of a torus)
- The **n-th homology group** is the quotient:

$$H_n(X) = \frac{Z_n}{B_n} = \frac{\ker \partial_n}{\text{im } \partial_{n+1}}$$

What does H_n actually measure?

Algebraic equivalence:

- In the quotient group $H_n = Z_n/B_n$, two cycles are considered “equal” (homologous) if they differ by a boundary
- If you can slide one loop into another along a surface, their difference is the boundary of that surface piece, so they represent the **same element** in the homology group
- **Solving the $\mathbb{RP}^2$ problem**: While the 1^{st} Betti number is 0, the 1^{st} homology group is $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$
  The algebra perfectly captures the fact that a path must go around the projective plane *twice* to become a trivial boundary

Computing $H_1(\mathbb{RP}^2)$

The polygonal schema:

- We can flatten the hemisphere into a disk (a 2-dimensional face F) bounded by a single edge a folded and glued to itself in the same direction
- **The 1-chains (edges)**: The boundary of the edge a is a single vertex minus itself, so $\partial_1(a) = v - v = 0$.
- Thus, the loop a is a cycle: $Z_1 = \langle a \rangle \cong \mathbb{Z}$
- **The 2-chains (faces)**: The boundary of the solid face F consists of the edge a traversed *twice* in the same direction
- $\partial_2(F) = a + a = 2a$. Thus, the boundaries are the even multiples of a: $B_1 = \langle 2a \rangle \cong 2\mathbb{Z}$

The homology quotient:

$$H_1(\mathbb{RP}^2) = \frac{Z_1}{B_1} = \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2$$

The algebra “knows” that a path must wrap twice to bound a surface

Homotopy vs homology

Two different algebraic probes:

- By 1930, topology had two major tools to measure holes:
  1. **Homotopy (π_1)**: The fundamental group (tracking exact base-point loops)
  2. **Homology (H_1)**: The first homology group (tracking cycles modulo boundaries)
- **The trade-off**: The fundamental group π_1 is incredibly precise, but it is generally **non-Abelian** and notoriously difficult to compute
- Homology groups are strictly **Abelian** and can be computed via linear algebra (matrices)

The Hurewicz theorem (1935)

The bridge between the tools:

- Are these two measurements related? Yes
- Witold Hurewicz proved that the first homology group is simply the **Abelianisation** of the fundamental group
- Algebraically, you take the fundamental group and force all elements to commute by quotienting out its commutator subgroup:

$$H_1(X) \cong \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}$$

- Homology gives you the "commutative shadow" of the highly complex homotopy group

The Hurewicz theorem in action

The fundamental group of $\mathbb{RP}^2$:

- Consider a loop a connecting antipodal points on the sphere. This loop cannot be shrunk to a point.
- However, traversing the loop **twice** ($a * a$) creates a fully closed loop on the sphere, which can be slipped off the equator and shrunk to a point!
- Therefore, the fundamental group is generated by one element a, subject to the relation $a^2 = 1$ (the identity path):

$$\pi_1(\mathbb{RP}^2) = \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

Applying the theorem:

- Because $\mathbb{Z}/2\mathbb{Z}$ is already an Abelian (commutative) group, its commutator subgroup is trivial: $[\pi_1, \pi_1] = \{1\}$
- Hurewicz confirms: $H_1 \cong \pi_1/\{1\} \cong \mathbb{Z}/2\mathbb{Z}$

The Poincaré conjecture (1904)

The ultimate test of the machinery:

- At the end of his series on *Analysis Situs*, Poincaré posed a question that would haunt mathematics for a century
- He knew that any 2-dimensional surface with trivial homology ($H_1 = 0$) and trivial homotopy ($\pi_1 = 0$) is homeomorphic to the 2-sphere (S^2)
- **The conjecture**: Consider a closed 3-dimensional manifold M^3 If every loop can be shrunk to a point (i.e., it is simply connected, $\pi_1(M^3) = 0$), is M^3 necessarily homeomorphic to the 3-sphere S^3?
- He famously ended the paper with:

"But this question would lead us too far away"

Summary: The algebraic translation

A perfect dictionary:

- The era from Poincaré to Noether successfully reduced the qualitative geometry of space to pure algebra

topological space ⟶ groups

continuous deformations ⟶ group homomorphisms

- This translation guarantees that if two spaces have different homology or homotopy groups, it is **mathematically impossible** to continuously deform one into the other
- Space has been fully algebraicised. But the story of algebraic geometry still has one massive, abstract leap to make...

References

The main sources for this lecture are

- *Jean Dieudonné*, A History of Algebraic and Differential Topology, 1900-1960, Birkhäuser (1989)
- *John Stillwell*, Mathematics and Its History, third edition, Springer (2010)
- *Colin McLarty*, "Emmy Noether's 'Set Theoretic' Topology: From Dedekind to the Rise of Functors," in *The Architecture of Modern Mathematics* (2006)

For students wanting to see the formal modern definitions built from these historical roots, *Allen Hatcher's* Algebraic Topology (2001) is the absolute standard reference

The Evolution of Space: Lecture 9

Syllabus:

- The algebraic synthesis:
  From varieties to sheaves

The return to polynomials

A parallel universe:

- While Poincaré and Noether were stretching space into topological rubber, another group of mathematicians was returning to the rigid roots of Descartes and analytical geometry
- **Algebraic geometry** studies the shapes defined by the roots of polynomial equations (e.g., $y^2 = x^3 + ax + b$)
- In the 19^{th} and early 20^{th} centuries (the Italian school), these shapes—called **algebraic varieties**—were mostly studied over the complex numbers $\mathbb{C}$
- But as abstract algebra grew, mathematicians wanted to study these geometric shapes over *any* field (finite fields, rational numbers, etc.)

The Italian school

Intuition over rigour:

- Between 1885 and 1935, mathematicians like Guido Castelnuovo, Federigo Enriques, and Francesco Severi achieved brilliant results
- Their crowning achievement was the **classification of algebraic surfaces** (2-dimensional varieties)
- **The crisis**: Their methods were highly intuitive, relying on visual geometric reasoning over $\mathbb{C}$. As mathematics demanded stricter formalisation, their proofs were found to contain significant gaps
- Their geometric intuition collapsed when pushed to higher dimensions or fields of positive characteristic. A rigorous algebraic foundation was desperately needed

The Zariski topology

A radically different space:

- In 1944, Oscar Zariski introduced a new topological structure to study these varieties, completely independent of the Euclidean metric
- **The affine space**: Let $\mathbb{A}^n$ be the n-dimensional space over a field k (simply the set of n-tuples of elements in k)
- **The definition**: In the **Zariski topology** on $\mathbb{A}^n$, the *closed sets* are precisely the algebraic varieties (the zero-sets of polynomials)
- Because a non-zero polynomial only vanishes on a "thin" lower-dimensional subset (like a curve in a plane), its complement is necessarily **massive**
- Consequently, any two non-empty open sets intersect!

A highly non-Hausdorff world

Losing classical intuition:

- In standard (Hausdorff) topology, any two distinct points can be separated by disjoint open neighbourhoods
- In the Zariski topology, this is **impossible**
- Points are "glued" together algebraically You cannot isolate a point without describing it as the exact intersection of polynomial roots
- This topology is incredibly "coarse", but it perfectly captures the algebraic rigidity of the space

Open sets are almost everywhere

Geometry becomes ring theory

The coordinate ring:

- How do we study a variety V? We study the functions that live on it
- Let $I(V)$ be the ideal of all polynomials that vanish on V
  The **coordinate ring** is $A(V) = k[x_1, \dots, x_n]/I(V)$
- **Hilbert's Nullstellensatz** established a perfect dictionary:

  $$\text{Points of the variety } V \iff \text{Maximal ideals of the ring } A(V)$$

- To understand the geometry of V, one simply studies the algebraic structure of the ring $A(V)$

Example: The affine line

The simplest space:

- Consider the 1-dimensional affine line over the complex numbers, $V = \mathbb{A}^1_{\mathbb{C}}$
- What functions live on the entire line? Only polynomials in one variable
- Therefore, the coordinate ring is simply $A(\mathbb{A}^1) = \mathbb{C}[x]$
- By the Nullstellensatz, every point $a \in \mathbb{A}^1$ corresponds exactly to the maximal ideal generated by $(x - a)$
- **Therefore, the geometry of the line is entirely encoded in the algebra of polynomials**

The local-to-global problem

The limit of global functions:

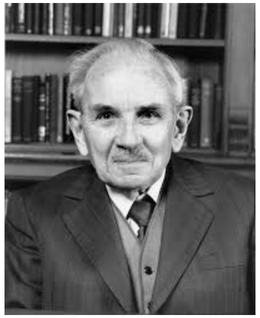
- A coordinate ring $A(V)$ consists of **global** polynomials defined everywhere on V
- But geometry is often local! What about rational functions (fractions of polynomials) that are perfectly well-defined in an open set U, but blow up (divide by zero) elsewhere?
- We need a way to mathematically track data (functions, differential forms, vector fields) that are defined locally, and understand how they glue together globally
- The solution came from a completely unexpected place

Example: The need for local functions

Functions with poles:

- Returning to the affine line $\mathbb{A}^1$, consider the function $f(x) = \frac{1}{x}$
- This is a rational function, so it is **not** an element of the global coordinate ring $\mathbb{C}[x]$
- However, from a geometric perspective, $f(x)$ is perfectly smooth and well-behaved everywhere except at the origin $x = 0$
- If we restrict our space to the massive Zariski open set $U = \mathbb{A}^1 \setminus \{0\}$, the "correct" ring of functions for U must include $\frac{1}{x}$
- We need a mathematical tool to assign different rings of functions to different open sets

The birth of sheaves (1945)



Jean Leray

Oflag XVII-A:

- During WWII, the French mathematician **Jean Leray** was held in a German prisoner-of-war camp
- To avoid doing applied math for the war effort, he turned to pure algebraic topology
- He invented the concept of a **sheaf** (*faisceau*) to systematically track how local homological data varies over a space
- Henri Cartan and Jean-Pierre Serre later adapted Leray's sheaves to complex analysis and algebraic geometry

What is a sheaf?

Tracking local data:

- A **presheaf** assigns to every open set U a ring of functions $\mathscr{O}(U)$ (e.g., all functions valid on U)
- It includes **restriction maps**: if $V \subset U$, you can restrict a function from U down to V
- **The gluing axiom**: A presheaf becomes a **sheaf** if local data that agrees on overlaps can be uniquely "glued" together to form global data

Think of a sheaf as a sophisticated mathematical bookkeeping system that perfectly manages the relationship between the local and the global

Example: The structure sheaf $\mathscr{O}$

Equipping the line:

- We can define a specific sheaf of rings, called the structure sheaf $\mathscr{O}$, on our affine line $\mathbb{A}^1$
- On the whole space, it assigns the global polynomials: $\mathscr{O}(\mathbb{A}^1) = \mathbb{C}[x]$
- On the open set $U = \mathbb{A}^1 \setminus \{0\}$, it assigns the Laurent polynomials: $\mathscr{O}(U) = \mathbb{C}[x, x^{-1}]$
- If we have a local function on $\mathbb{A}^1 \setminus \{0\}$ and another on $\mathbb{A}^1 \setminus \{1\}$ that agree exactly on the overlap $\mathbb{A}^1 \setminus \{0,1\}$, the sheaf gluing axiom guarantees they combine into a single, valid function

Summary: The ringed space

A new definition of space:

- By the 1950s, algebraic geometry had a new foundational object: the **ringed space**
- A space is no longer just a set of points with a topology
- A space is a pair $(X, \mathscr{O}_X)$:
  1. X: A topological space (often with the coarse Zariski topology)
  2. $\mathscr{O}_X$: A **sheaf of rings** dictating exactly what "functions" are allowed on any open patch of X
- This set the stage for Alexander Grothendieck, who would ask: if the sheaf $\mathscr{O}_X$ tells us everything about the geometry, do we even need the underlying points of X to be classical geometric dots?

References

The main sources for this lecture are

- *Jean Dieudonné*, History of Algebraic Geometry, Wadsworth Advanced Books (1985)
- *Robin Hartshorne*, Algebraic Geometry, Springer (1977) (Chapter II for the modern formalisation)
- *Igor Shafarevich*, Basic Algebraic Geometry 1, Springer (2013)
- *Jeremy Gray*, A History of Abstract Algebra, Springer (2018)

For Master's students, Dieudonné's historical overview is excellent for understanding the philosophical motivations behind the mid-20$^{\text{th}}$-century algebraic geometry rewrite

The Evolution of Space: Lecture 10

Syllabus:

- The Grothendieck revolution:
  Schemes, primes, and infinitesimals

Beyond classical varieties

The limit of the Nullstellensatz:

- In the previous lesson, we saw that the classical points of a variety correspond exactly to the **maximal ideals** of its coordinate ring
- In the 1950s, Alexander Grothendieck posed a radical philosophical question about the nature of points
- Why should we artificially restrict our geometry only to maximal ideals? What happens if we treat **every prime ideal** as a point?
- By dropping this restriction, Grothendieck expanded the universe of geometry to include completely new, bizarre types of spatial entities

Remind that an ideal P of a commutative ring R is *prime* when, (1) if $a, b \in R$ are such that their product $ab \in P$ then $a \in P$ or $b \in P$, and (2) $P \neq R$

The prime spectrum

Redefining space:

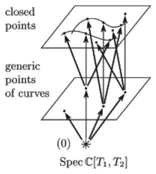
- Let R be any commutative ring. We define its **spectrum**, denoted $\mathrm{Spec}(R)$, as the set of all prime ideals of R
- We equip $\mathrm{Spec}(R)$ with the Zariski topology, where closed sets are defined as $\{V_I \colon I \text{ is an ideal of } R\}$ with V_I the set of prime ideals containing I
- We also equip it with a natural structure sheaf $\mathscr{O}$, making it a fully functioning ringed space
- **The paradigm shift**: The ring R now completely dictates the space
  We can plug *any* commutative ring into Spec to generate a geometry

Example: The affine line revisited

The emergence of the generic point:

- Let us return to the complex affine line, looking at $\mathrm{Spec}(\mathbb{C}[x])$
- The maximal ideals are $(x - a)$ for any $a \in \mathbb{C}$
  These are the standard, classical geometric dots
- But the zero ideal (0) is also a prime ideal
  Actually, the only prime ideals in $\mathrm{Spec}(\mathbb{C}[x])$ are (0) and $(x - a)$
- It corresponds to a completely new point, called the **generic point**
- This point does not sit at any specific coordinate
  Its topological closure is the *entire line*: it represents the "generic" behaviour of the entire space rolled into a single point

Arithmetic becomes geometry



The primes as a geometric space

The ultimate unification:

- What happens if we take a ring that has nothing to do with polynomials, like the integers $\mathbb{Z}$?
- $\mathrm{Spec}(\mathbb{Z})$ consists of the prime ideals $(2), (3), (5), \ldots$, plus the generic point (0)
- Prime numbers are now literally **geometric points** on a spatial curve

- This revolutionary insight allowed mathematicians to use the heavy machinery of geometry and topology to solve ancient problems in number theory (eventually leading to the proof of Fermat's last theorem)

Nilpotents and fuzzy points

The dual numbers:

- Consider the ring $R = \mathbb{C}[x]/(x^2)$
  This ring contains an element x where $x \neq 0$, but $x^2 = 0$ (a **nilpotent**)
- Classically, the only solution to $x^2 = 0$ is $x = 0$
  Classical geometry sees this space as a single, standard point
- But in $\mathrm{Spec}(R)$, the function x does not vanish everywhere, even though its square does
- Geometrically, this represents a point with **infinitesimal thickness**. It is a point equipped with a tangent vector, capturing "first-order" fuzziness

Example: The geometry of $x^2 = 0$

Intersections and tangency:

- Consider intersecting the parabola $y = x^2$ with the line $y = c$
- If $c > 0$, we get two distinct classical points: $x = \pm\sqrt{c}$
- If we lower the line to $c = 0$, the intersection is given by $x^2 = 0$
  Classically, the two points collapse into just one point: $(0, 0)$
- Classical geometry forgets that this was a tangency!

The scheme remembers:

- The scheme $\mathrm{Spec}(\mathbb{C}[x]/(x^2))$ retains the "ghost" of the second point
- Evaluate a polynomial $f(x)$ on this space. Because $x^2 = 0$, the Taylor expansion truncates instantly:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots \implies f(x) = f(0) + f'(0)x$$

- A "function" on this fuzzy point is exactly its **value** at the origin plus its **first derivative**. It is a point equipped with a tangent vector

The birth of the scheme

Gluing the spectra together:

- An **affine scheme** is simply a ringed space isomorphic to $\mathrm{Spec}(R)$ for some commutative ring R
- A **scheme** is a ringed space that locally looks like an affine scheme
- Just as Riemann built manifolds by gluing together flat Euclidean patches, Grothendieck built schemes by gluing together prime spectra of rings
- This framework is so robust that it completely subsumed all previous definitions of algebraic varieties

Summary: The triumph of algebra

The new rules of space:

- By identifying points with prime ideals, geometry absorbs number theory
- By allowing rings with nilpotents, geometry absorbs differential calculus (infinitesimals)
- The scheme represents the ultimate victory of algebra over classical spatial intuition
- The final step, however, is to realise that we do not even need the topological space X or the prime ideals to do geometry
  We only need the **relationships** between the objects. . .

References

The main sources for this lecture are

- *David Mumford*, The Red Book of Varieties and Schemes, Springer (1999)
- *Jean Dieudonné*, History of Algebraic Geometry, Wadsworth Advanced Books (1985)
- *Robin Hartshorne*, Algebraic Geometry, Springer (1977)
- *Ravi Vakil*, The Rising Sea: Foundations of Algebraic Geometry, Princeton University Press (2025)

For Master's students, Mumford's "Red Book" is highly recommended as it masterfully explains the geometric intuition behind the intense abstraction of schemes and the related concepts

The Evolution of Space: Lecture 11

Syllabus:

- Categorical probes:
  The Yoneda perspective

The end of reductionism

A new way to know an object:

- Since the ancient Greeks, mathematics relied on **reductionism**: to understand an object, you break it down into its smallest internal pieces (points, elements, prime ideals)
- In the 1950s, category theory proposed a radical alternative borrowed intuitively from quantum physics
- You cannot see the internal structure of a subatomic particle
  You can only deduce its nature by throwing other particles at it and measuring the **interactions**
- Category theory applies this to space: do not look inside the space X
  Instead, measure how every other space T interacts with X

Probing a space

The functor of points:

- Let X be our target space (a topological space, a scheme, etc.)
- Let T be a "test" space. We probe X by looking at the set of all possible morphisms (continuous maps, scheme morphisms) from T into X
- We denote this set of arrows as $\mathrm{Hom}(T, X)$
- As we vary the test space T across all possible spaces in our category, we generate a massive functor h_X:

$$h_X(T) = \mathrm{Hom}(T, X)$$

- This functor h_X is essentially the "shadow" or "profile" of X projected onto the rest of the mathematical universe

The birth of categories (1945)

A language for topology:

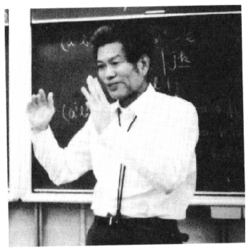
- Category theory was born in 1945 from the work of **Samuel Eilenberg** and **Saunders Mac Lane** in their seminal paper *General Theory of Natural Equivalences*
- Their original goal was not to rewrite the foundations of mathematics, but simply to formalise how algebraic topology translates topological spaces into algebraic groups
- To rigorously define a "natural transformation" between these topological dictionaries, they first had to define the dictionaries themselves (**functors**)
- And to define functors, they had to rigorously define the source and target environments (**categories**)

The categorical trinity

The basic dictionary:

- **Category**: A collection of objects (dots) and morphisms (arrows between dots). Arrows can be composed, and every object has an "identity" arrow doing nothing
- **Functor**: A mapping from one category to another. It maps dots to dots and arrows to arrows, perfectly preserving the network of compositions (think of it as embedding a wireframe model into a new space)
- **Natural transformation**: A mapping *between* functors. If two functors F and G are two different wireframe embeddings of the same category, a natural transformation is the smooth, structure-preserving "sliding" of the F-wireframe into the G-wireframe

Nobuo Yoneda (1930–1996)



米田信夫

Nobuo Yoneda

A fateful encounter in Paris:

- In 1954, a young Japanese mathematician named **Nobuo Yoneda** met Saunders Mac Lane at a train station in Paris
- Before his train departed, Yoneda sketched out a profoundly deep categorical lemma to Mac Lane
- Mac Lane later formalised and named it the **Yoneda lemma**. It has since become the cornerstone of modern algebraic geometry and category theory
- It answers the ultimate question: does the external "profile" h_X capture *everything* about X?

The Yoneda lemma

The mathematical statement:

- Let $\mathscr{C}$ be a category, X an object in $\mathscr{C}$, and h_X its associated probe functor ($h_X(T) = \text{Hom}(T, X)$)
- Let F be *any* other functor from $\mathscr{C}$ to the category of Sets
- The Yoneda lemma states that the set of natural transformations (mappings between functors) from h_X to F is in perfect bijection with the elements of $F(X)$:

$$\text{Nat}(h_X, F) \cong F(X)$$

While the equation looks intimidating, its philosophical consequence is what shatters our classical definition of space

The philosophical consequence

The Yoneda embedding:

- What happens if we replace the arbitrary functor F with another probe functor h_Y (representing a space Y)?
- The lemma yields: $\text{Nat}(h_X, h_Y) \cong h_Y(X) = \text{Hom}(X, Y)$
- **The ultimate corollary**: If $h_X \cong h_Y$ (meaning X and Y have the exact same interactions with all test spaces T), then $X \cong Y$
- **Translation**: An object is *completely uniquely determined* by its network of relationships

Example: Knowing a space by its probes

Probing the circle:

- Imagine our target space X is a circle (S^1) concealed in a dark box
- We throw different "test" spaces T at it and record the valid continuous maps $\text{Hom}(T, X)$:
  - If $T = \{*\}$ (a single point), $\text{Hom}(\{*\}, S^1)$ gives us the exact set of **points** on the circle
  - If $T = [0,1]$ (a line segment), $\text{Hom}([0,1], S^1)$ gives us all the **paths** and loops on the circle
  - If $T =$ a disk, we find all the ways to wrap a 2D surface over it
- The Yoneda lemma guarantees that if a mystery shape Y returns the exact same dataset of points, paths, and wraps as S^1 for *every* test space T, then Y **must** be a circle

Summary: The death of the internal

Space as a network:

- Thanks to Yoneda, we no longer need to know what a space is "made of"
- If two spaces look identical from the perspective of all incoming arrows, they are mathematically identical
- **Identity is purely relational**
- This frees Grothendieck to construct geometries where the objects are not sets of points at all, but rather pure, abstract nodes in a web of arrows
- This sets the stage for the final abstraction: the **topos**

References

The main sources for this lecture are

- *Saunders Mac Lane*, Categories for the Working Mathematician, 2nd edition, Springer (1978)
- *Emily Riehl*, Category Theory in Context, Dover (2017)
- *Jean-Pierre Marquis*, From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory, Springer (2009)

For Master's students, Riehl's modern exposition is fantastic for demonstrating how the Yoneda lemma functions not just as an abstract theorem, but as a practical philosophical tool

The Evolution of Space: Lecture 12

Syllabus:

- Topoi and the ultimate abstraction: Grothendieck sites and logical universes

The failure of the Zariski topology

Too coarse for calculus:

- By the late 1950s, Alexander Grothendieck and his school had successfully rewritten algebraic geometry using schemes and sheaves
- However, they hit a wall: the Zariski topology has too few open sets
- In standard calculus, the **inverse function theorem** guarantees that if a function's derivative is non-zero at a point, we can zoom in to a small enough open neighbourhood where the function is strictly invertible
- In the Zariski topology, open sets are so large and overlapping that you can almost never isolate a small enough patch to make this work
- To count solutions to polynomial equations over finite fields (the famous **Weil conjectures**), Grothendieck needed a "finer" topology
  But no classical topology worked

Grothendieck topologies (1962)

Replacing subsets with arrows:

- Grothendieck applied the Yoneda philosophy: do not look inside the space
- Classically, an open cover of a space X is a union of subsets $U_i \subset X$
- Grothendieck redefined a "covering" as a collection of **arrows** (morphisms) $U_i \to X$ that jointly cover X, even if the U_i are not subsets
- A category equipped with a valid system of these categorical coverings is called a **site**
- **The paradigm shift**: You no longer need an underlying set of points to define a topology. You only need a category of arrows and a rule for what constitutes a "cover"

Classical topology as a site

The standard open cover:

- Take a standard topological space X (like a sphere)
- Instead of viewing open sets U as collections of points, we view them as **objects** in a category
- The **morphisms** are the inclusion maps: $U \hookrightarrow V$ if U is a subset of V
- A "covering" of V is defined purely categorically: it is a family of incoming arrows $\{U_i \hookrightarrow V\}$ such that they jointly build V
- This translates the classical set-theoretic union ($\bigcup U_i = V$) entirely into the language of arrows

Breaking the subset barrier

The étale topology:

- Now, Grothendieck's genius:
  the arrows in our cover *do not need to be inclusions*
- Imagine a circle X
  Now imagine a second circle U wrapped twice around X (like a spring)
  The projection map $U \to X$ locally looks identical to X, but U is **not** a subset of X
- Grothendieck declared that families of these "local homeomorphisms" (**étale morphisms**) form a perfectly valid covering of X
- We can now "cover" a space with shapes that are larger and topologically more complex than the space itself

The topos

The ultimate spatial object:

- Once you have a site (a categorical topology), you can define **sheaves** on it, just as Leray did for classical spaces
- Grothendieck defined a **topos** as the category of all sheaves on a site
- And then, he made a staggering conceptual leap:

  "The topos itself is the space"
- The underlying site is merely a temporary scaffolding
- The true geometric invariant is the massive category of sheaves. Two completely different sites can generate the exact same topos, meaning they are geometrically identical

Space without points

The final liberation:

- A topos is a "generalised space" that has completely freed itself from the tyranny of set theory and points
- In classical topology, a space is a set of points. In a topos, there are no points; there are only the structural relationships (sheaves) that *would* have existed if the points were there
- **Geometry becomes purely relational**
- This allowed Grothendieck to construct spatial structures that behave perfectly like continuous manifolds, even when the underlying data is completely discrete (like prime numbers)

Example: The topos of graphs

A universe without standard points:

- To build intuition, consider the category of all **directed graphs**
- This entire category forms a topos. It behaves exactly like the classical universe of sets, but with a twist
- In standard set theory, the fundamental building block is an isolated, featureless **point**
- In graphs, the fundamental building blocks are **nodes and edges**
- If you do algebra, topology, or logic inside the topos of graphs, every construction you make is inherently aware of connectivity. You are doing mathematics in a universe where "space" fundamentally means "network"

The logical revolution (1970)

Geometry becomes logic:

- In 1970, F. William Lawvere and Myles Tierney discovered a profound secondary property of Grothendieck's topoi
- A topos is not just a generalised space
  It is a **universe of mathematical discourse**
- Every topos possesses its own internal logical rules. In most topoi, the logic is **intuitionistic** (the law of excluded middle, $P \vee \neg P$, is false)
- Geometry, topology, and mathematical logic are completely unified. Studying the geometry of a topos is mathematically identical to studying the logical rules of a specific universe

Epilogue: The journey of space (I)

The classical era (measurement):

- **Antiquity (Euclid)**:
  Space is an absolute, rigid void hosting physical shapes
  It is the practical geometry of the builder and the astronomer
- **The 17th century (Descartes)**:
  Space is translated into a quantifiable grid of coordinates ($\mathbb{R}^n$)
  Geometry becomes algebra

Epilogue: The journey of space (II)

The structural era (invariants):

- **The 19th century (Riemann & Klein)**:
  The rigid void is destroyed
  Space becomes an intrinsic, flexible manifold defined by a metric tensor, or the invariant structure of a transformation group
- **The early 20th century (Poincaré & Noether)**:
  The metric is discarded
  Space is a continuous topological medium classified entirely by algebraic holes (homology)

Epilogue: The journey of space (III)

The categorical era (relations):

- **The late 20th century (Grothendieck & Lawvere)**:
  The "point" is finally abandoned
  Space is a categorical universe of logic and relations (a topos)
- The mathematical focus completely shifts from **ontology** (what space is made of) to **epistemology** (how a space interacts with others)

Geometry began as the practical measurement of the earth
Over 2,500 years, it evolved into the pure, abstract study of structural relationships

References

The main sources for this lecture are

- *Colin McLarty*, The Uses and Abuses of the History of Topos Theory, *British Journal for the Philosophy of Science* (1990)
- *Jean-Pierre Marquis*, From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory, Springer (2009)
- *F. William Lawvere* and *Stephen H. Schanuel*, Conceptual Mathematics, Cambridge University Press (2009)

For Master's students, McLarty's historical account provides the best non-technical bridge into why Grothendieck was forced to invent the topos to solve the Weil conjectures

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