History of Mathematics

The Journey of Mathematics

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Corso di Laurea Magistrale in Matematica

a.a. 2026/27

Syllabus:

Bureaucracy
- Program & texts
- Examination
- Timing
- Questions

Let's start!
- Methodology
- Prehistory

Introduction

This course offers a comprehensive exploration of the evolution of mathematical thought and practice.

The first part provides a concise overview of the development of Mathematics from prehistory to the close of the 20$^{\text{th}}$ century, highlighting major advancements and shifts in mathematical understanding.

The second and third parts are monographic, allowing for a deeper investigation into specific areas of mathematical history. The themes and instructors for these two parts may vary from year to year, reflecting current research and scholarly interests.

Program

The program of the whole course is then
1. 32 hours: **General part** (these slides)
2. 24 hours: **Monographic part A**
3. 24 hours: **Monographic part B**

Since the monographic parts vary from year to year, their programs will be detailed at their beginnings

Program

The program of the general part illustrates the most significant periods from the mathematical point of view:

1. *Egyptian and Mesopotamian Mathematics*
2. *Greek Mathematics*
3. *Ancient Asian Mathematics*
4. *The Islamic Golden Age*
5. *Middle Ages and Renaissance*
6. *Mathematics in the 17th century: The dawn of modernity*
7. *Mathematics in the 18th century: The age of Euler*
8. *Mathematics in the 19th century: The age of revolution*
9. *The 20th century in Mathematics*

Texts

All the slides are available at the course website:

`https://marcobenini.me/lectures/history-of-mathematics/`

Also, at the end of each lesson, references to resources which may be of interest to those interested in learning more, will be provided. While the content of the slides is **mandatory**, looking at the references is **optional**.

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)

are the official reference texts for the first part of the course.

Furthermore, links to the original sources are available on the course web page, as well as other relevant material.

Examination

There are two ways to take the examination:

1. Writing and discussing an essay
2. Taking three partial assignments

If the student chooses the first way, the examination consists of writing a paper on a topic agreed upon with one of the course instructors, chosen from those covered during the teaching period.

The paper will be assessed during an oral discussion considering its historical accuracy, the ability to correlate the topic with its developmental context and with other related areas of mathematics, and the quality of its explanation, particularly for a technical but non-specialist audience, and for an interested but non-technical audience.

Alternatively, at the student's choice, three intermediate written tests will be conducted during the course period, aimed at verifying the learning and comprehension of the three parts into which the course is divided.

Timing

The schedule of lessons is fixed and not modifiable.

Please note that lessons typically begin at the official start time and conclude 15 minutes before the official end time, with no breaks within.

Intermediate assignments will be integrated into the course schedule.

Students will have the flexibility to choose a date for these assignments after completing the relevant section of the course.

Questions

Questions are welcome. Please, do not hesitate to ask questions when you do not understand something during a lesson.

Questions could be asked also before the start of a lesson, or after the end. Another possibility is to ask questions by email: if so, please write to the address `marco.benini@uninsubria.it` specifying your name, the course, and the question. Please, use your official email from `uninsubria`.

There are no office hours in this course: students have to fix an appointment. Please, do so only if you really think there is no other way to solve your problem: although I am usually available during the course term, when I am not teaching it is often the case that I am not in University, so use this opportunity as your last resource.
Online appointments are always possible and encouraged.

Let's start!

The history of mathematics

Today's plan:

- **Course aims and approach**:
  Understanding the unique perspective of this course
- **Sources in the history of Mathematics**:
  How do we know what we know?
- **Acknowledging the narrative focus**:
  Understanding the course's scope and limitations
- **Major eras in Mathematics**:
  A chronological roadmap
- **Mathematics in pre-history**:
  Looking back before written records

Course aims and approach

Our goal: To understand **how** and **why** mathematical concepts evolved, connecting past breakthroughs to modern understanding.

Navigating the evolution of mathematical ideas:

- This course offers a **journey** through the history of Mathematics, specifically designed **for mathematicians**

Course aims and approach

- It is not intended as a general historical survey, but rather a focused exploration with a particular emphasis:
  - Highlighting the **groundbreaking results** of each period and their revolutionary impact
  - Providing **concrete references** to key achievements, foundational texts, and the researchers behind them
  - Whenever possible, drawing **direct support from primary sources** to convey the authentic mathematical voice of the time
  - An illustration style that emphasises **what is important within a continuous line of narration**, rather than an exhaustive list
  - By design, the course is **not exhaustive**; it focuses on depth and insight over comprehensive coverage of every detail
  - We prioritise a **sound and coherent narrative** that illuminates the development of mathematical thought over a purely chronological or encyclopedic description

Sources in the history of mathematics

Where does our knowledge come from?

The historian's challenge: Reconstructing the past

- Unlike mathematics itself, history relies heavily on **evidence**
- For the history of mathematics, this evidence isn't always straightforward
- We often piece together fragments from diverse origins

Sources in the history of mathematics

Primary sources: The voices of the past

- **Original mathematical texts**: Treatises, letters, notebooks, ancient tablets (e.g., Babylonian clay tablets, Egyptian papyri, Greek manuscripts)
- Provide direct insight into how mathematicians of the time thought, formulated problems, and presented their solutions
- Often challenging to interpret due to language, notation differences, and lost context
- **Archaeological evidence**: Tools, structures, artefacts that suggest mathematical understanding (e.g., bone carvings for counting)

Sources in the history of mathematics

Secondary sources: Interpretations and syntheses

- **Historical studies and commentaries**: Books and articles written by modern historians of mathematics
- These works analyse, translate, and contextualise primary sources, often providing critical perspectives and identifying broader trends
- Crucial for understanding complex developments, but always subject to scholarly debate and new discoveries

Sources in the history of mathematics

Challenges in interpretation:

- **Lost knowledge**: Many texts are lost, damaged, or incomplete
- **Translation difficulties**: Nuances of ancient languages and specialised terminology can be hard to render precisely
- **Anachronism**: The danger of viewing past ideas through modern lenses, misinterpreting their original meaning or significance
  We strive to avoid this

A western-centric journey

The scope of our exploration:

- While aiming for a broad historical understanding, we will primarily follow a **Western-centric narrative** of mathematical development
- This means our journey will largely trace the lineage of mathematical ideas from ancient Greece through the European Renaissance, and into modern Western academia

A western-centric journey

Why this focus?

- **Academic tradition**: This aligns with the historical development of university curricula in which the course itself is situated
- **Source availability**: While growing, readily accessible primary and robust secondary sources for a comprehensive global history are still more concentrated in the Western tradition for a single course
- **Direct lineage**: The mathematics taught in most contemporary Western curricula (and thus relevant to "mathematics for mathematicians") directly descends from this specific historical trajectory

A western-centric journey

- It is absolutely vital to acknowledge that mathematical thought is a **universal human endeavour** with vibrant, independent, and profoundly influential traditions across the globe
- Significant contributions arose from:
  - **Ancient civilisations**: Egypt, Mesopotamia (as we'll see)
  - **China**: Independent development of algebra, geometry, number theory
  - **India**: Place-value system, zero, foundations of trigonometry and calculus
  - **The Islamic Golden Age**: Preservation and advancement of Greek and Indian knowledge, development of algebra, algorithms, and non-Euclidean geometry precursors
  - **Mesoamerica**: Sophisticated calendar systems and numerical notation
- We will touch upon these as they inform or intersect with the Western line of development, but a full global history would require dedicated courses

A western-centric journey

We will speak about ancient civilisations; we will sketch Indian and Chinese contributions, and we will treat the influence of Islamic mathematics on the European scene

We will sketch a bit the Japanese contribution: the reason for this is that Japanese mathematics is a somewhat unique case of an advanced but still independent mathematical tradition

A necessary limitation, not a value judgement: Our focus is a pragmatic choice for this course's scope. It in no way diminishes the immense value and ingenuity of mathematical traditions worldwide, which represent humanity's collective intellectual heritage

Major eras in the history of mathematics

Dividing the past: A conventional periodisation

- History is continuous, but periodisation helps us organise and understand major shifts in mathematical thought and methodology
- These periods are broad and can overlap, especially when considering global developments

Major eras in the history of mathematics

A chronological roadmap

1. **Pre-historic Mathematics** (before ~3000 BCE):
   - Early concepts of number, counting, basic measurement, recognition of patterns, fundamental figures
   - Evidence from archaeological discoveries (e.g., tally sticks)
2. **Ancient Mathematics** (~3000 BCE – 600 CE):
   - **Mesopotamia & Egypt**: Practical arithmetic, geometry, early algebra (solving equations)
   - **Greece**: Rise of deductive reasoning, axiomatic systems (Euclid), rigorous geometry, number theory (Pythagoreans, Diophantus), early calculus ideas (Archimedes)
   - **China & India** (Contemporary developments): Independent advancements in numeration (place value, zero), algebra, trigonometry

Major eras in the history of mathematics

3. **Medieval Mathematics** (~600 CE – 1400 CE):
   - **Islamic Golden Age**: Preservation of Greek and Indian texts, major advancements in algebra, trigonometry, algorithms, number theory, and early analytical geometry. Transmission to Europe
   - **Europe**: Limited until late period, re-engagement with classical texts via Islamic scholars
4. **Renaissance and early modern Mathematics** (~1400 CE – 1700 CE):
   - Revival of European mathematics. Solutions to cubic/quartic equations, logarithms, analytic geometry (Descartes, Fermat), foundations of probability theory
   - **Calculus**: Independent development by Newton and Leibniz

Major eras in the history of mathematics

5. **Modern Mathematics** (~1700 CE – 20th century):
   - Extensive development of calculus, mechanics, analysis (Euler, Lagrange, Laplace) and the raise of structural problems
   - **19th century:** Rigorisation of analysis, abstract algebra, non-Euclidean geometries, foundations of set theory
6. **Contemporary Mathematics** (20th century – present):
   - Abstraction, axiomatisation, foundational crisis, mathematical logic, computability, topology, functional analysis
   - Explosion of new fields, interdisciplinary connections (physics, computer science), grand problem-solving

Mathematics before written records

The dawn of mathematical thought: A challenge for historians

- Our understanding of "pre-historic" mathematics relies on archaeological and anthropological evidence, as written records did not yet exist
- This period spans from the earliest human cognitive development (tens of thousands of years ago) up to the emergence of complex civilisations with writing systems (~3000 BCE)
- While we cannot speak of formal "mathematics", we can infer the development of fundamental mathematical concepts

Mathematics before written records

Evidence of early mathematical thinking:

- **Counting and numerosity**:
  - Evidence of tally marks on bones (e.g., the **Ishango Bone**, ~20,000 BCE), suggesting counting and perhaps prime number awareness
  - One-to-one correspondence for comparison (e.g., stones for sheep)
  - Development of number words and early counting systems

The Ishango bone (~20,000 BCE)

Mathematics before written records

Evidence of early mathematical thinking:

- **Measurement**:
  - Basic understanding of length, weight, volume, and time (cycles of the moon and seasons) for practical needs (building, trading, agriculture)
  - Use of body parts (cubit, foot) as early units
- **Geometry and spatial reasoning**:
  - Recognition of basic geometric patterns in nature (spirals, symmetries)
  - Practical geometry for shelter construction, tool making, and early art
  - Understanding of spatial relationships, balance, and proportions

Mathematics before written records

The emergence of abstract concepts:

- The crucial step from “three deer” to the abstract concept of “three”
- This abstraction allowed for the development of counting systems independent of the objects being counted

From concrete needs to abstract ideas: Pre-historic mathematical activity, driven by practical necessity, laid the cognitive groundwork for the formal systems that would emerge with the advent of writing and complex societies, representing the fundamental human impulse to quantify, order, and understand the world

References

Three classical references are:

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Dirk J. Struik* A Concise History of Mathematics, Fourth revised edition, Dover Publications (1987)

The course is based on them

Also, very good references to authors and ideas can be found in the Stanford Encyclopedia of Philosophy: `https://plato.stanford.edu/`

History of Mathematics: Lecture 2

Syllabus:

Egyptian and Mesopotamian Mathematics

Egyptian mathematics

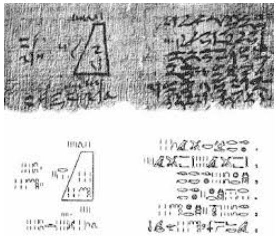
Mathematics was an essential tool for practical needs such as:

- **Constructions**: pyramids, temples, obelisks
- **Administration**: taxes, land surveying, labour management
- **Astronomy** and **calendars**

Our primary understanding comes from:

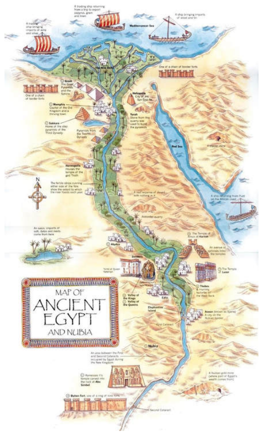
- the *Moscow papyrus* (~1800 BCE):
  - a collection of 25 problems, including the volume of a truncated pyramid
- the *Rhind papyrus* (~1650 BCE):
  - a collection of 85 problems, including arithmetic, geometry, and algebra

We will primarily focus on the Rhind papyrus



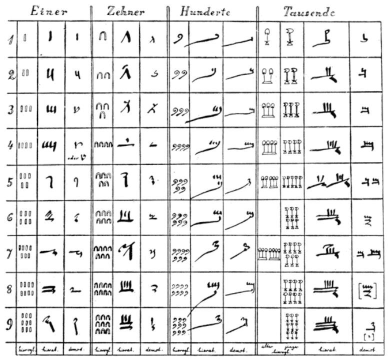
(fragment of the Moscow papyrus)

Key periods and mathematical relevance



- **Early Dynastic Period** (~3100–2686 BCE)
  - Early development of writing, calendars, and basic administration
- **Old Kingdom** (~2686–2181 BCE)
  - Period of pyramid building (e.g., Giza); implies advanced surveying and geometry
- **Middle Kingdom** (~2055–1650 BCE)
  - Era of the *Moscow papyrus* (~1800 BCE)
  - Flourishing of administrative systems and early mathematical texts
- **Second Intermediate Period** (~1650–1550 BCE)
  - Period of the *Rhind papyrus* (~1650 BCE)
- **New Kingdom** (~1550–1070 BCE)
  - Continued use of mathematical practices; building of grand temples

Egyptian numerals



By *Florian Cajori*, A History of Mathematical Notations (1928)

Two numbering systems:

- *Hieroglyphic numerals*: the formal way of writing, for ceremonial use
- *Hieratic script*: the everyday script for scribes, more cursive

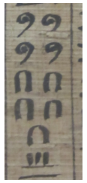
Both are **additive systems**

Egyptian numerals

Hieroglyphic numerals:

1	a single stroke
10	a hobble for cattle
100	a coil of rope
1,000	a lotus plant
10,000	a finger
100,000	a tadpole or frog
1,000,000	a god

453



4,622

1,333,330

Photographs by Cynthia J. Huffman (Pittsburgh State University)
from *An Ancient Egyptian Mathematical Photo Album*,
Mathematical Association of America

Egyptian arithmetic

The fundamental operation is addition.

Multiplication was computed by *duplication*: to calculate $a \times b$, duplicate b until the number of duplications becomes greater or equal than a. Then[a] $a \times b$ is the sum of those duplications whose sum gives a.

[a]We tried to write the rule as found in the translation of the Rhind papyrus. Admittedly, the example is much clearer.

23×17

1	17
2	34
4	68
8	136
16	272

Since $32 > 23$ and $23 = 16 + 4 + 2 + 1$,
$23 \times 17 = 272 + 68 + 34 + 17 = 391$

Egyptian arithmetic

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Why? In contemporary terms, the *binary representation* of 23 is 10111 thus

$$\begin{aligned} 23 \times 17 &= (2^4 \times 1 + 2^2 \times 1 + 2^1 \times 1 + 2^0 \times 1) \times 17 \\ &= 2^4 \times 17 + 2^2 \times 17 + 2^1 \times 17 + 2^0 \times 17 \\ &= 272 + 68 + 34 + 17 = 391 \ . \end{aligned}$$

Egyptian arithmetic

Division is *reverse multiplication*: consider $234/13$

13	1
26	2
52	4
104	8
208	16

Since $234 < 2 \times 208 = 416$ and
$234 = 208 + 26$ thus
$234/13 = 16 + 2 = 18$

$$\begin{aligned} 234 &= 208 + 26 \\ &= 2^4 \times 13 + 2^1 \times 13 \\ &= (2^4 + 2^1) \times 13 \\ &= (16+2) \times 13 \\ &= 18 \times 13 \end{aligned}$$

Hence $234/13 = 18$.

Egyptian arithmetic

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Hence $234/13 = 18$.

When a division resulted in a remainder, representing "parts of a whole" was crucial for practical problems like sharing goods or dividing land.
Since the Egyptians did not use common fractions like $3/5$, they developed a unique system: **unit fractions**.

Unit fractions

What are unit fractions?

- A fraction of the form $1/n$, where n is a positive integer
- Represented by placing a symbol resembling an open mouth (hieroglyph for 'part') over the numeral. E.g. $1/3$ is

- The only exception was $2/3$, which had its own special symbol:

Unit fractions

Why unit fractions?

- Egyptians did not use fractions like $3/4$ or $5/7$ as single entities
- Any non-unit fraction was expressed as a *sum* of distinct unit fractions
  - e.g., $3/5$ would be written as $1/2 + 1/10$
  - The decomposition required all denominators to be distinct
- This system allowed for precise division and sharing of resources (e.g., loaves of bread, grain)

(A fragment of the Rhind papyrus)

Unit fractions

Example 2.1. Sharing loaves (*Rhind papyrus*, problem 3)

- **Problem**: Divide 6 loaves equally among 10 men
  (Modern: $6/10 = 3/5$ per man)
- **Egyptian solution**: Each man receives $1/2 + 1/10$ of a loaf
  - Check: $1/2 + 1/10 = 5/10 + 1/10 = 6/10$
- This approach made sense for physical division:
  - Give each man half a loaf. (5 loaves used)
  - Divide the remaining 1 loaf among 10 men, giving $1/10$ to each

The $2/n$ table (*Rhind papyrus*)

- To aid in these decompositions, the Rhind papyrus begins with a large table listing the unit fraction expansions for $2/n$ for odd n from 5 to 101
- Example: $2/7 = 1/4 + 1/28$
- The methods for finding these decompositions are still debated, but show considerable ingenuity

Algebraic equations

Example 2.2. (*Rhind papyrus*, problem 24)

- **Problem**: A quantity (*aha*) and its $1/7$ added together become 19.
  What is the quantity? (Modern: find x such that $x + \frac{1}{7}x = 19$)

Algebraic equations

Example 2.2. (*Rhind papyrus*, problem 24)

- **Problem**: A quantity (*aha*) and its $1/7$ added together become 19.
  What is the quantity? (Modern: find x such that $x + \frac{1}{7}x = 19$)

Pose $x = 7$. Then

1	7
1/7	1
	8

False position method:
$x/7 = 19/8$

Algebraic equations

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1	7
1/7	1
	8

False position method: $x/7 = 19/8$

Calculate 19/8:

1	8	
2	16	*
1/2	4	
1/4	2	*
1/8	1	*
$2+\frac{1}{4}+\frac{1}{8}$	19	

Thus $\frac{1}{7}x = 2 + \frac{1}{4} + \frac{1}{8}$

Algebraic equations

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- **Problem**: A quantity (*aha*) and its 1/7 added together become 19. What is the quantity? (Modern: find x such that $x + \frac{1}{7}x = 19$)

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1	8	
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1/2	4	
1/4	2	*
1/8	1	*
$2+\frac{1}{4}+\frac{1}{8}$	19	

Thus $\frac{1}{7}x = 2 + \frac{1}{4} + \frac{1}{8}$

Indeed,

1	$2+\frac{1}{4}+\frac{1}{8}$
2	$4+\frac{1}{2}+\frac{1}{4}$
4	$9+\frac{1}{2}$
7	$16+\frac{1}{2}+\frac{1}{8}$
1/7	$2+\frac{1}{4}+\frac{1}{8}$
	19

Hence $x = 16 + \frac{1}{2} + \frac{1}{8}$

Geometry

Example 2.3. (*Rhind papyrus*, problem 48)

- **Problem**: Compare the area of a circle and of its circumscribing square.

Geometry

Example 2.3. (*Rhind papyrus*, problem 48)

- **Problem**: Compare the area of a circle and of its circumscribing square.

Egyptian method for circle area

Rule: Take 1/9 off the diameter, and square the result.

For a circle of diameter $D = 9$:

$$\text{Area}_{\text{circle}} = \left(D - \frac{D}{9}\right)^2 = 64$$

Ahmes found the area of the circumscribing square of side 9:

$$\text{Area}_{\text{square}} = 9^2 = 81$$

Geometry

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Deriving the implied π

In modern terms

$$\text{Area}_{\text{circle}} = \pi r^2$$

$$64 = \pi \left(\frac{9}{2}\right)^2$$

$$\pi = \frac{64 \times 4}{81}$$

$$\pi = \frac{256}{81} \approx 3.16\ldots$$

Geometry

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$$\pi = \frac{64 \times 4}{81}$$

$$\pi = \frac{256}{81} \approx 3.16\ldots$$

A remarkably good approximation: This value of $\pi \approx 3.16$ is surprisingly accurate for its time (compare to Archimedes' later bounds, for example)

Geometry

Example 2.4. (*Moscow papyrus*, problem 14)

- **Problem**: The volume of a truncated pyramid (*frustum*)

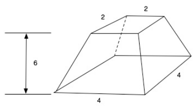

Diagram of a truncated pyramid

- This corresponds to:
  - Height (h) = 6 units
  - Side of lower base (a) = 4 units
  - Side of upper base (b) = 2 units

Geometry

Example 2.4. (*Moscow papyrus*, problem 14)

- **Problem**: The volume of a truncated pyramid (*frustum*)

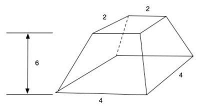

Diagram of a truncated pyramid

- This corresponds to:
  - Height (h) = 6 units
  - Side of lower base (a) = 4 units
  - Side of upper base (b) = 2 units

The Egyptian method

1. Square the base side: $4^2 = 16$
2. Square the top side: $2^2 = 4$
3. Multiply the base by the top side: $4 \times 2 = 8$
4. Add these three results: $16 + 4 + 8 = 28$
5. Take one third of the height: $\frac{1}{3} \times 6 = 2$
6. Multiply the sum from step 4 by the result from step 5: $28 \times 2 = 56$

Geometry

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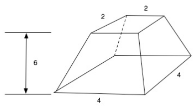

Diagram of a truncated pyramid

- This corresponds to:
  - Height (h) = 6 units
  - Side of lower base (a) = 4 units
  - Side of upper base (b) = 2 units

The calculated volume is:

56 cubic units

The Egyptian method

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3. Multiply the base by the top side: $4 \times 2 = 8$
4. Add these three results: $16 + 4 + 8 = 28$
5. Take one third of the height: $\frac{1}{3} \times 6 = 2$
6. Multiply the sum from step 4 by the result from step 5: $28 \times 2 = 56$

Geometry

Modern formula for a square frustum: The volume V of a square frustum with height h and base sides a and b is:

$$V = \frac{h}{3}(a^2 + ab + b^2)$$

Substituting the values from the problem ($h = 6$, $a = 4$, $b = 2$):

$$\begin{aligned} V &= \frac{6}{3}(4^2 + 4 \cdot 2 + 2^2) \\ &= 2(16 + 8 + 4) \\ &= 2 \cdot 28 = 56 \end{aligned}$$

Significance: The Egyptian calculation yields the *exact modern formula*. The method by which they derived this formula is unknown, showcasing remarkable sophistication

The character of Egyptian mathematics

Key characteristics:

- *Highly practical & utilitarian*: Developed as a tool for administration, construction (pyramids, temples), land surveying, and calendar keeping. Not abstract mathematics for its own sake
- *Algorithmic & procedural*: Focus on step-by-step methods and "recipes" to solve specific problems, rather than generalised theories or proofs
- *Additive arithmetic*: Multiplication and division relied heavily on doubling and halving (binary-like approach)
- *Unit fraction system*: A distinctive feature where all fractions (except 2/3) were expressed as sums of distinct reciprocals ($1/n$). This led to complex manipulation techniques (e.g., 2/n table)

The character of Egyptian mathematics

Major achievements:

- *Efficient number system*: Their hieroglyphic and hieratic systems allowed for large numbers and practical calculations
- *Sophisticated arithmetic techniques*: Mastery of operations with integers and unit fractions, enabling complex financial and resource management
- *Advanced geometry*:
  - Area of a circle with a remarkably good approximation of $\pi \approx 3.16$
  - Exact formula for the volume of a truncated square pyramid (frustum), demonstrating advanced spatial reasoning
- *Early record keeping*: The Rhind and Moscow papyri provide invaluable insights into their mathematical thought process and problem-solving

A foundation for future developments: Egyptian mathematics was robust and effective for their needs, laying groundwork for later traditions, even if their methods differed significantly from modern axiomatic approaches

Mesopotamian mathematics

Key themes to explore:

- The revolutionary *Sexagesimal* (Base-60) *Positional System*
- Sophisticated algebraic problem-solving
- Deep insights into number theory (Pythagorean triples)
- Remarkable geometric precision (e.g., $\sqrt{2}$)

Geographic and chronological context

- "Land between the rivers" (Tigris and Euphrates)
- Cradle of civilisation: Sumer, Babylon, Assyria
- Abundant clay (source of cuneiform tablets)

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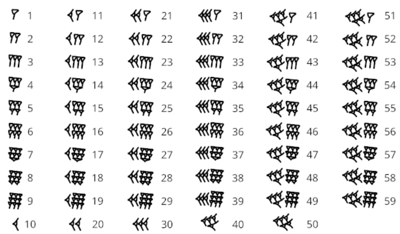
- **Sumerian Period** (~4000–2000 BCE)
  - Earliest writing (cuneiform) and number systems
  - Basic accounting, metrology
- **Babylonian Period** (~2000–1600 BCE)
  - **Golden age** of mathematics
  - Most of our mathematical tablets (especially, *Plimpton 322* and *YBC 7289*) date from this era
  - Focus on solving practical and theoretical problems
- **Later Periods** (Kassite, Neo-Babylonian, . . .)
  - Continued mathematical activity, especially in astronomy

The revolutionary sexagesimal system

Positional system:

- Similar to ours, the *position* of a digit determines its value
- Uses combinations of two wedge symbols
- Numbers within each position go from 1 to 59



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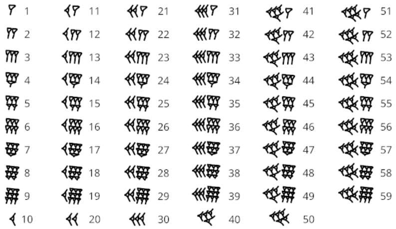

Example 2.5. $1:2:3_{60}$

$$\begin{aligned} 1:2:3_{60} &= 1\times 60^2 + 2\times 60^1 + 3\times 60^0 \\ &= 1\times 3600 + 2\times 60 + 3\times 1 \\ &= 3600 + 120 + 3 = 3723_{10} \end{aligned}$$

Note: They did not have a true ‘zero’ placeholder (initially) or a sexagesimal point. Context often inferred magnitude

The revolutionary sexagesimal system

Strength in fractions:

- Allows for convenient representation of many fractions:

$$\begin{aligned} 1/2 &= 30/60 = 0,30_{60} \\ 1/3 &= 20/60 = 0,20_{60} \\ 1/4 &= 15/60 = 0,15_{60} \\ 1/5 &= 12/60 = 0,12_{60} \\ 1/6 &= 10/60 = 0,10_{60} \\ 1/10 &= \ 6/60 \ = 0,06_{60} \end{aligned}$$

The revolutionary sexagesimal system

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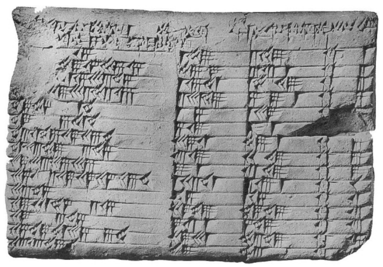
Why is this powerful?

- 60 is highly composite (divisors: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60)
- Thus many common fractions have finite sexagesimal representations
- Made division and handling of fractions much simpler than in base-10 or Egyptian unit fractions

Plimpton 322: A Pythagorean puzzle



Plimpton 322 (~1800 BCE)

What is it?

- A Babylonian clay tablet from the Old Babylonian period
- Discovered in southern Iraq in the early 20th century
- Currently housed at Columbia University
- Perhaps the most famous and debated mathematical tablet

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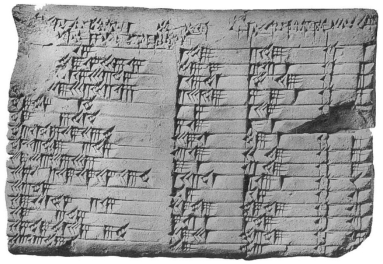

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Its content:

- A table of 15 rows and 4 columns of numbers written in sexagesimal
- The numbers are remarkably large, indicating sophisticated calculations
- Primarily interpreted as a list of *Pythagorean triples*

Plimpton 322: Pythagorean triples

Reconstructed columns of Plimpton 322:

	width w	diagonal d	number
1.9834...	119	169	1
1.94916...	3367	11521 (4825)	2
1.9188...	4601	6649	3
1.88635...	12709	18541	4
1.81501...	65	97	5
1.78519...	319	481	6
1.71988...	2291	3541	7
1.6928...	799	1249	8
1.64267...	541 (481)	769	9
1.58612...	4961	8161	10
1.5625...	45	75	11
1.48942...	1679	2929	12
1.34002...	25921 (161)	289	13
1.43024...	177	3229	14
1.38716...	56	53 (106)	15

Plimpton 322: Pythagorean triples

Check for row 5:

- $w = 65,\ d = 97$
- Let $b^2 = d^2 - w^2$
- $b^2 = 97^2 - 65^2 = 5184$
- $b = \sqrt{5184} = 72$
- So, the triple is $(65, 72, 97)$

What are *Pythagorean Triples*?

- Sets of three positive integers (a, b, c) such that $a^2 + b^2 = c^2$
- Examples: $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$

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Babylonian interest:

- Advanced understanding of geometry, including right triangles
- These triples could be used for:
  - Land surveying and demarcation
  - Construction (ensuring right angles)
  - Astronomical calculations (though less direct evidence)
- This knowledge predates Pythagoras by over a millennium

How were the triples generated?

Hypothesis 1: Generating *primitive triples* (Euclid's formula)

- Modern formula for generating primitive Pythagorean triples: If m, n are coprime integers, $m > n$, and one is even, one is odd, then:

$$a = m^2 - n^2$$
$$b = 2mn$$
$$c = m^2 + n^2$$

- While the formula itself wasn't known in this form, the Babylonians might have implicitly used such a method
- Example: For $(m, n) = (16, 9)$, the triple is $(175, 288, 337)$. For row 13 $(161, 240, 289)$, this corresponds to $m = 15$, $n = 8$

How were the triples generated?

A more likely method for Plimpton 322:

- Generation using *reciprocal pairs* and *regular numbers*
- Babylonian scribes used tables of reciprocals for division
- A *regular number* is one whose prime factors are only 2, 3, or 5 (the prime divisors of 60). Their reciprocals have finite sexagesimal expansions

How were the triples generated?

Hypothesis 2: Van der Waerden's *reciprocal pair* method (simplified)

- Consider a pair of regular numbers (u, v) such that u/v is the ratio of two sides of the right triangle
- The sides of the right triangle (a, b, c) can be generated from u and v as:

$$a = u^2 - v^2$$
$$b = 2uv$$
$$c = u^2 + v^2$$

  (This is essentially Euclid's formula, but the point is how they **found** the u, v pairs, which were often from reciprocal tables)
- The crucial point is that all the u/v ratios that generate the Plimpton 322 rows are **regular sexagesimal numbers**
- This suggests a systematic way of finding such pairs for generating the table, possibly related to solving problems of finding sides of rectangles given area and diagonal, or similar algebraic problems

Significance and interpretations

Why is Plimpton 322 so important?

- *Pythagorean knowledge before Pythagoras:* Provides undeniable evidence that the Babylonians knew and systematically generated Pythagorean triples over a thousand years before Pythagoras
- *Advanced number theory:* Shows a sophisticated understanding of integer properties, prime factorisation (in the context of regular numbers), and how to generate specific sets of numbers
- *Algorithmic approach:* Whether for geometry or number theory, the tablet represents a systematic, algorithmic approach to problem-solving
- *Potential for early trigonometry:* Some scholars argue that the tablet might have served as a kind of "secant table" or a proto-trigonometric table for specific angles in right triangles

Significance and interpretations

Ongoing debates:

- Was it a list of "Pythagorean triples" for geometric construction, a teaching tool, or a number theory research tablet?
- Exactly which method was used to generate the rows?
- What was the exact purpose of the first column?
  Observe how it can be generated as $\left(\frac{d}{c}\right)^2$ where c is the length of the other side of the right triangle beside d and w

Plimpton 322 demonstrates the profound mathematical ingenuity of the old Babylonian scribes

YBC 7289: Approximating $\sqrt{2}$

(YBC 7289 — ~1800-1600 BCE)

What is it?

- A small Babylonian clay tablet from the Old Babylonian period
- Discovered in southern Iraq
- Housed at Yale University
- One of the most famous examples of Mesopotamian numerical approximation

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Its content:

- A square with diagonals drawn
- Numbers written in sexagesimal (base-60)
- The numbers represent the side and diagonal of the square

The Babylonian approximation

The numbers on the tablet:

- Side of the square: 30
- Diagonal of the square: 42:25:35

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- 42:25:35 in base 60 means: $42 + \frac{25}{60} + \frac{35}{60^2} \approx 42.42638889$

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$$\frac{42:25:35}{30} \approx \frac{42.42638889}{30} \approx 1.414212963$$

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- This level of precision is remarkable for the time

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Implications:

- The Babylonians had a sophisticated algorithm (though we don't know the exact method) for approximating square roots
- They understood the concept of irrational numbers

YBC 7289: How did they find it?

- The exact method is unknown, but several possibilities exist:
  - *Geometric methods*: Repeated averaging of sides and diagonals of squares or rectangles
  - *Algebraic methods*: An iterative algorithm similar to the Babylonian method for solving quadratic equations:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right)$$

    where S is the number whose square root is being approximated (in this case, $S = 2$)
- The Babylonians were skilled in both geometry and algebra, so either approach is plausible
- Regardless of the method, the result shows their computational power

YBC 7289: Significance

- *Accurate approximation of an irrational number*: Demonstrates a deep understanding of numbers, including those that cannot be expressed as a simple fraction
- *Computational skill*: Shows their ability to perform complex calculations with remarkable precision
- *Influence on later mathematics*: This level of accuracy influenced later Greek mathematics and beyond

YBC 7289 is a testament to the sophisticated numerical techniques of the old Babylonian period

References

The lecture is based on:

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)

Further reading:

- *Eleanor Robson*, Neither Sherlock Holmes nor Babylon: A Reassessment of Plimpton 322, *Historia Mathematica* 28 (2001), 167–206

Just for pleasure:

- *Denis Guedj*, Il teorema del pappagallo, Longanesi, 2000

History of Mathematics: Lecture 3

Syllabus:

Greek Mathematics: *The birth of deduction*

Greek mathematics

From "How" to "Why"

A fundamental paradigm shift

Egyptian & Mesopotamian Mathematics:

- Practical, utilitarian, algorithmic
- Focus on procedures and numerical solutions (the "How")
- Examples: Doubling for multiplication, $2/n$ tables, volume of frustum

Greek Mathematics (Starting ~6th century BCE):

- Theoretical, abstract, philosophical
- Emphasis on **deductive reasoning** and **formal proof** (the "Why")
- Mathematics as an axiomatic system

Philosophy and the pursuit of truth

Mathematics as a branch of Philosophy:

- Unlike earlier civilisations, Greeks sought **universal and eternal truths**
- Numbers and geometric forms were seen as embodying perfect, unchanging realities
- This quest for truth led to the demand for **rigorous justification** for mathematical statements

Philosophy and the pursuit of truth

Key philosophers influencing mathematics:

- **Plato** (~428–347 BCE):
  - "Let no one ignorant of geometry enter here" (inscription at the Academy)
  - Believed mathematical objects (*forms*) were more real than physical ones
  - Emphasised abstract thought over sensory experience
- **Aristotle** (~384–322 BCE):
  - Systematised logic (*syllogisms*) and the structure of arguments
  - His ideas on deduction influenced Euclid's axiomatic method

Plato and Aristotle
(from Raphael's "School of Athens")

The legacy of inquiry: This philosophical drive for certainty and general principles profoundly shaped the development of Western mathematics

Sources and early schools

How do we know? (sources):

- Unlike Egyptian/Mesopotamian papyri/tablets, much of our knowledge comes from **later commentators**:
    - *Proclus* (~412–485 CE): Wrote "Commentary on the First Book of Euclid's Elements", preserving much historical detail
    - *Pappus of Alexandria* (~4^{th} century CE): Compiled earlier works
- **Euclid's Elements**: The monumental summary of earlier Greek geometry and number theory
- Fragments and direct works of later major figures (e.g., Archimedes, Apollonius, Diophantus)

Sources and early schools

Early centres of mathematical thought:

- **Ionian School** (Miletus, ~6^{th} century BCE):
    - Focus on natural philosophy and rational inquiry
    - Figures: **Thales** (often considered the first to seek proofs)
- **Pythagorean School** (Croton, ~6^{th}–5^{th} century BCE):
    - Emphasised the mystical and philosophical nature of numbers
    - Major discoveries: Pythagorean theorem (proof), **incommensurability**
- **Platonic Academy** (Athens, ~4^{th} century BCE):
    - Geometry seen as essential for intellectual training
    - Many future mathematicians trained here

Thales of Miletus: The dawn of proof

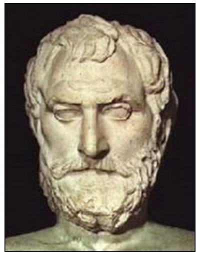
Who was Thales? (~624–546 BCE)

- From Miletus, Ionia (modern Turkey)
- Considered the first philosopher and scientist in the Western tradition
- Often regarded as the **first mathematician to seek proofs** rather than just observing rules or procedures
- Travelled to Egypt, where he likely learned practical geometry

The revolutionary idea of proof:

- Previous civilisations used geometry for practical tasks
- Thales (and the Greeks) asked: *Why is this true*?
- This led to the demand for logical, deductive arguments based on definitions and axioms

Thales of Miletus: The dawn of proof



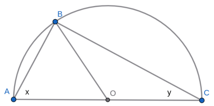
Thales of Miletus

Key geometric theorems attributed to Thales:

- An angle inscribed in a semicircle is a right angle
- The base angles of an isosceles triangle are equal
- Vertically opposite angles are equal
- Two triangles are congruent if they have two angles and the included side equal
- A circle is bisected by its diameter

Thales: Example of deductive reasoning

Theorem 3.1. *An angle inscribed in a semicircle is a right angle (Euclid, Elements, Book III, Proposition 31)*



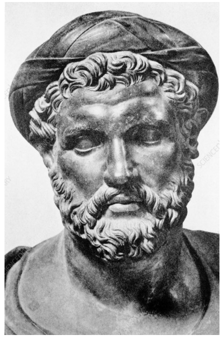
- Let A, B, C be points on a circle, with AC being a diameter; let O be the centre of the circle
- Draw radius OB
- Triangles AOB and BOC are isosceles
- Let $\angle OAB = x$ and $\angle OCB = y$
- Then $\angle OBA = x$ and $\angle OBC = y$
- The sum of angles in $\triangle ABC$ is 180°: $x + (x + y) + y = 180°$, thus $2x + 2y = 180° \implies 2(x + y) = 180°$, hence $x + y = 90°$
- Since $\angle ABC = x + y$, then $\angle ABC = 90°$. Q.E.D.

This demonstrates the shift from empirical observation to logical derivation from basic principles.

The Pythagoreans

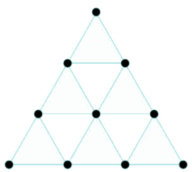
The Pythagorean Brotherhood (~570–495 BCE):

- Founded by **Pythagoras of Samos** in Croton (Southern Italy)
- More than just a school: a secret, quasi-religious brotherhood
- Believed that **numbers were the essence of all things**
- Motto: "*all is number*"
- Focused on purifying the soul through study, especially of music and mathematics



Pythagoras of Samos

The Pythagoreans



The Tetractys

Number mysticism and harmony:

- Explored relationships between numbers and the natural world
- Discovered that musical harmony could be expressed by simple numerical ratios (e.g., octave 1:2, fifth 2:3, fourth 3:4)
- Associated numbers with specific qualities:
  - 1: point, reason
  - 2: line, opinion
  - 3: surface, wisdom
  - 4: solid, justice
- The **Tetractys** (sum of $1 + 2 + 3 + 4 = 10$) was considered sacred

The Pythagorean theorem

The theorem:

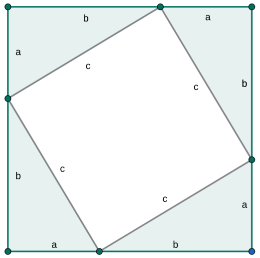
- In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides ($a^2 + b^2 = c^2$)
- This relationship was **known empirically** to Babylonians and Egyptians over 1000 years earlier (e.g., Plimpton 322)

Pythagorean contribution:

- The Pythagoreans are widely credited with providing the **first formal, deductive proof** of the theorem
- This exemplifies the core Greek contribution: moving from practical knowledge to abstract, proven truth

The Pythagorean theorem

- Consider a large square of side $(a+b)$
- Inside it, arrange four identical right triangles with sides a, b, c
- The central unshaded area will be a square of side c (area c^2)
- Since the total area of the large square and the four triangles is constant, $c^2 = a^2 + b^2$
- This visual proof (or a variant) is often attributed to Pythagoreans



Rearrangement proof of $a^2 + b^2 = c^2$

A milestone in deductive thought: This proof established a powerful precedent for mathematical reasoning, moving beyond calculation to rigorous demonstration

The crisis of incommensurability

The "axiom" of "all is number":

- The Pythagoreans believed all quantities could be expressed as ratios of integers (rational numbers)
- This was fundamental to their entire philosophical system

The discovery of incommensurability:

- Applying the Pythagorean Theorem to a simple square of side 1:

$$1^2 + 1^2 = d^2 \implies d^2 = 2 \implies d = \sqrt{2}$$

- Legend attributes the discovery to Hippasus of Metapontum
- He proved that the diagonal of a square is **incommensurable** with its side
- In other words, $\sqrt{2}$ **cannot be expressed as a ratio of two integers**, i.e., $\sqrt{2}$ is irrational

The crisis of incommensurability

Proof: $\sqrt{2}$ is irrational, by contradiction:

- Assume $\sqrt{2} = p/q$, where p, q are integers, $q \neq 0$, and p/q is in lowest terms (no common factors)
- Squaring both sides: $2 = p^2/q^2 \implies 2q^2 = p^2$
- This implies p^2 is an even number. If p^2 is even, then p must also be even
- So, we can write $p = 2k$ for some integer k
- Substitute $p = 2k$ back into $2q^2 = p^2$: $2q^2 = (2k)^2 \implies 2q^2 = 4k^2 \implies q^2 = 2k^2$
- This implies q^2 is an even number. If q^2 is even, then q must also be even
- Contradiction! We assumed p and q had no common factors, but we showed both must be even (thus having a common factor of 2)
- Therefore, our initial assumption ($\sqrt{2}$ is rational) must be false. $\sqrt{2}$ is irrational Q.E.D.

The crisis of incommensurability

Consequences for Greek Mathematics:

- Shattered the "all is number" philosophy based solely on rational numbers
- Led to a profound conceptual crisis for their understanding of "number"
- Forced a shift in focus from arithmetic to **geometry**, where quantities (lengths, areas) could be compared and reasoned about even if their numerical ratio was unknown
- *Geometric algebra*: Algebraic problems were often solved using geometric constructions

Euclid and the Elements

Euclid of Alexandria (~325–265 BCE):

- Lived and taught in Alexandria, Egypt (Ptolemaic kingdom)
- Little is known about his life; he's often referred to as "the author of the Elements"
- His work synthesised over 300 years of Greek mathematical thought

Euclid of Alexandria

Euclid and the Elements

Page from a medieval manuscript of Euclid's Elements

The "Elements" (~300 BCE):

- Not a record of new discoveries (mostly), but a **systematic compilation and organisation** of existing mathematical knowledge
- Comprises **13 books**, primarily on geometry and number theory
- Became the definitive textbook for geometry for over 2000 years, influencing countless thinkers

The axiomatic method

Structure of the Elements:

- Euclid began with a small set of self-evident truths:
  - *Definitions*: Clearly defined terms (e.g., point, line, surface)
  - *Postulates (axioms for geometry)*: Fundamental assumptions about geometric constructions that are taken as true without proof (e.g., "A straight line may be drawn between any two points")
  - *Common notions (axioms for general math)*: General truths applicable to all quantities (e.g., "Things which are equal to the same thing are also equal to one another")
- From these initial statements, he logically **deduced** a vast system of propositions (theorems)
- Each proposition is proven using only definitions, postulates, common notions, and previously proven propositions

Euclid's enduring legacy

Why is this Revolutionary?

- **Rigour & certainty**: Established a benchmark for mathematical rigour, aiming for absolute certainty
- **Logical structure**: Provided a systematic way to organise mathematical knowledge
- **Foundation for all Mathematics**: This deductive, axiomatic approach became the model for mathematical inquiry for millennia, from Newton's *Principia* to modern set theory
- **Independent of experience**: Mathematical truth became independent of physical measurement or empirical observation

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Why is this Revolutionary?

- **Rigour & certainty**: Established a benchmark for mathematical rigour, aiming for absolute certainty
- **Logical structure**: Provided a systematic way to organise mathematical knowledge
- **Foundation for all Mathematics**: This deductive, axiomatic approach became the model for mathematical inquiry for millennia, from Newton's *Principia* to modern set theory
- **Independent of experience**: Mathematical truth became independent of physical measurement or empirical observation

The **parallel postulate**: One of Euclid's postulates (often phrased as "Through a point not on a given line, exactly one line parallel to the given line can be drawn") remained controversial, leading to the development of **non-Euclidean geometries** in the 19th century.
This shows the power of examining axioms

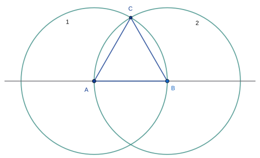
Euclid's Elements

Beyond basic geometry:

- **Book I**: Fundamental geometry (points, lines, triangles, parallels)
- **Book II**: Geometric algebra (e.g., constructions for solving quadratic equations, geometric identities)
- **Book V**: Eudoxus's theory of proportion (rigorous treatment of ratios, including incommensurables, solving the Pythagorean crisis)
- **Books VII-IX**: Number theory (primes, greatest common divisor algorithm, perfect numbers, proof of infinitely many primes)
- **Book X**: Classification of incommensurable magnitudes
- **Book XIII**: Construction of the five *Platonic solids*

Euclid's Elements

Proof (Proposition 1, Book I):
Problem: To construct an equilateral triangle on a given finite straight line



Construction sketch:

1. With centre A, draw a circle with radius AB
2. With centre B, draw a circle with radius BA
3. Let C be one of the points where the circles intersect
4. Draw lines AC and BC

Euclid's Elements

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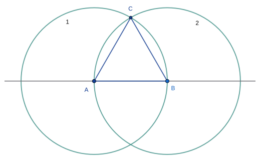
Problem: To construct an equilateral triangle on a given finite straight line



Construction sketch:

1. With centre A, draw a circle with radius AB
2. With centre B, draw a circle with radius BA
3. Let C be one of the points where the circles intersect
4. Draw lines AC and BC

Proof sketch:

- All radii of a circle are equal. So $AC = AB$ (from circle 1) and $BC = AB$ (from circle 2)
- By Common Notion 1 ("Things which are equal to the same thing are also equal to one another"), $AC = BC$
- Therefore, triangle ABC is equilateral Q.E.D.

References

The lesson is based on:

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Dirk J. Struik* A Concise History of Mathematics, Fourth revised edition, Dover Publications (1987)

Also, very good references to authors and ideas can be found in the Stanford Encyclopedia of Philosophy: `https://plato.stanford.edu/`

The fundamental reference text is *Euclid*, The Thirteen Books of the Elements, three volumes, Dover (1956)

Finally, just for the fun of the readers, we signal *Paolo Alessandrini*, Matematica Rock, Hoepli (2019), which also speaks about the Pythagorean theme of mathematics and music

History of Mathematics: Lecture 4

Syllabus:

Greek Mathematics: *Beyond Euclid*

- Dealing with the infinite (Zeno's paradoxes)
- Rigorous treatment of magnitude and proportion
- Early "Calculus" (Method of Exhaustion)
- The genius of Archimedes
- Conic sections and their importance
- The Hellenistic period and the eventual decline

Zeno's paradoxes: Confronting the infinite

Zeno of Elea (~490–430 BCE):

- A pre-Socratic philosopher, student of Parmenides
- Known for his paradoxes concerning motion and plurality
- While philosophical, they had profound implications for the mathematical understanding of **infinity** and **continuity**

Zeno's paradoxes: Confronting the infinite

The dichotomy paradox:

- To reach a destination, one must first cover half the distance
- Then, one must cover half of the remaining distance
- This process continues infinitely
- *Conclusion* (Zeno): Therefore, motion is impossible, as one must complete an infinite number of tasks in a finite time

Achilles and the tortoise paradox:

- Achilles (the swift runner) races a tortoise, who is given a head start
- Achilles must first reach the tortoise's starting point
- By then, the tortoise has moved a little further
- Achilles must reach **that** new point, by which time the tortoise has moved. . . and so on, infinitely
- *Conclusion* (Zeno): Achilles can never catch the tortoise

Mathematical implication: Both paradoxes deal with an infinite geometric series: precisely, $1/2+1/4+1/8+\cdots=1$ in the dichotomy paradox

Zeno's paradoxes: Confronting the infinite

Impact on Greek thought:

- Highlighted the conceptual difficulties of infinitely divisible quantities
- Contributed to the Greek aversion to actual infinities in mathematical proofs, favouring potential infinities
- Prompted mathematicians (like Eudoxus) to develop rigorous methods to deal with magnitudes without relying on infinite processes directly

Eudoxus: Rigour in ratios and volumes

Eudoxus of Cnidus (~408–355 BCE):

- A brilliant mathematician and astronomer, student of Plato
- Resolved the crisis of incommensurability (from the Pythagoreans)
- Developed methods that were foundational for later Greek geometry and a precursor to integral calculus

Eudoxus: Rigour in ratios and volumes

1. The theory of proportion (Euclid's Book V):

- Provided a rigorous definition of proportion (ratio) that worked for **both commensurable and incommensurable magnitudes**
- This bypassed the need to define irrational "numbers" explicitly
- Definition: Four magnitudes A, B, C, D are proportional (*A is to B as C is to D*) if, for any (positive) integers m, n:
  - $mA > nB \implies mC > nD$
  - $mA = nB \implies mC = nD$
  - $mA < nB \implies mC < nD$
- This definition is essentially equivalent to Dedekind cuts from 19^{th} century real analysis

Eudoxus: Rigour in ratios and volumes

2. The method of exhaustion:

- A technique for finding the unknown area or volume of a curved figure by inscribing and circumscribing polygons (polyhedra) whose areas/volumes can be calculated
- The areas of the polygons "exhaust" the area of the curved figure as the number of sides increases
- Relies on an axiom (often attributed to Eudoxus, known as the Axiom of Archimedes): "Given two unequal magnitudes, if from the greater there is subtracted a magnitude greater than its half, and from the remainder a magnitude greater than its half, and so on, there will at some time remain a magnitude less than the lesser of the given magnitudes"
- **Significance**: This was the closest the Greeks came to integral calculus, avoiding the paradoxes of infinity by focusing on what can be "exhausted"
- Used by Euclid to prove theorems about circles and spheres (e.g., area of circle proportional to square of diameter)

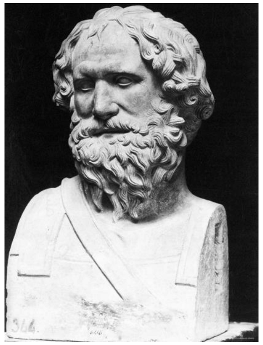
Eudoxus: Rigour in ratios and volumes

Impact: Eudoxus's work provided the tools for rigorous geometry in the face of incommensurability and laid the groundwork for Archimedes' later triumphs.

Archimedes of Syracuse

Archimedes of Syracuse (~287–212 BCE):

- Considered by many to be the **greatest mathematician of antiquity** and one of the greatest of all time
- Lived in Syracuse, Sicily, a Greek city-state
- Renowned not just for mathematics, but also for physics, engineering, and inventions (e.g., Archimedes' screw, levers, war machines)
- Legend: "Eureka!" (buoyancy principle) and "Give me a place to stand, and I will move the Earth" (lever principle)
- Killed during the Roman siege of Syracuse



Archimedes of Syracuse

Archimedes of Syracuse

His mathematical approach: Archimedes brilliantly applied the rigorous methods of Eudoxus (like the method of exhaustion) to solve an astonishing array of complex problems, often pushing these methods to their limits, bordering on what we now call calculus

The most accurate π of antiquity

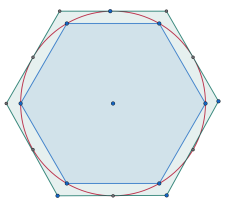
The problem:

- To find the ratio of a circle's circumference to its diameter, π
- Recall the Egyptian approximation: $\pi \approx (16/9)^2 \approx 3.16$

The method of perimeters

- He used a rigorous geometric approach based on the **method of exhaustion**
- Involved inscribing and circumscribing regular polygons within and around a circle
- As the number of sides of the polygons increases, their perimeters get closer and closer to the circumference of the circle

The most accurate π of antiquity



Inscribed and circumscribed polygons

Significance: This was the most accurate value of π for over 1000 years, derived purely from rigorous geometric methods, not empirical measurement

His calculation:

- Started with regular hexagons
- Systematically doubled the number of sides, going up to 96-sided polygons
- This involved complex calculations with square roots (often using rational approximations for them)
- Result:

$$3+\frac{10}{71}<\pi<3+\frac{10}{70}$$
$$3.140845<\pi<3.142857$$

(In modern terms, $223/71<\pi<22/7$)

Quadrature of the parabola

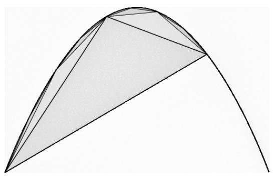
The problem:

- To find the area of a parabolic segment bounded by a chord
- This problem is typically solved with integral calculus today

Archimedes' method:

- Used the **method of exhaustion** (similar to Eudoxus')
- He filled the parabolic segment with an infinite sequence of triangles
- First, construct the largest triangle (with the given chord as its base)
- Then, in the remaining two parabolic segments, construct two more triangles, and so on
- He showed that the areas of these triangles form a geometric progression

Quadrature of the parabola



Quadrature of the parabola

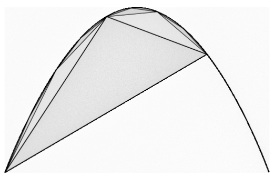
The result:

- He proved that the area of the parabolic segment is 4/3 the area of the first (largest) inscribed triangle
- The sum of the infinite series of triangle areas:

$$A+\frac{1}{4}A+\frac{1}{16}A+\cdots=A\left(1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\ldots\right)$$

  This is a geometric series with ratio $r=1/4$.
  Sum $=A\left(\frac{1}{1-1/4}\right)=A\left(\frac{1}{3/4}\right)=\frac{4}{3}A$
- Archimedes arrived at this result without the concept of limits or infinite series sums

Quadrature of the parabola



Quadrature of the parabola

Precursor to calculus: This demonstrates his profound ingenuity in handling infinite processes rigorously, centuries before Newton and Leibniz

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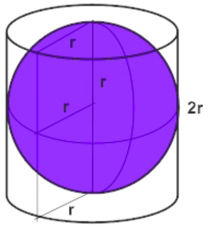
Volumes of solids and “the Method”

Volumes and surface areas of solids:

- Archimedes determined the formulas for the volume and surface area of many solids, notably the **sphere** and the **cylinder**
- His proof for the volume ($4/3\pi r^3$) and area ($4\pi r^2$) of a sphere was a crowning achievement
- He famously requested that his tombstone depict a sphere inscribed within a cylinder, along with the ratio 2 : 3 (Volume of Sphere : Volume of Cylinder = Area of Sphere : Area of Cylinder = 2 : 3)



Sphere inscribed in a cylinder

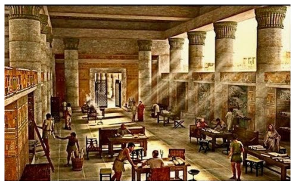
Volumes of solids and “the Method”

“The Method”: Discovery vs. proof

- Archimedes wrote a treatise called *The Method*, discovered in 1906 (the Archimedes Palimpsest)
- In it, he revealed how he **discovered** many of his results: by mechanical means, using principles of levers and centres of gravity
- For example, he imagined slicing shapes into infinitesimally thin sections and balancing them on a lever
- However, he then provided **rigorous proofs** (using the method of exhaustion) for these discoveries in separate works

A master of both heuristics and rigour: Archimedes showcased the dual nature of mathematics: the intuitive, experimental path to discovery, followed by the rigorous, logical path to proof

Mathematics in the Hellenistic era


The library of Alexandria

The Hellenistic period (~323–30 BCE):

- Followed Alexander the Great’s conquests, leading to the spread of Greek culture
- New intellectual centres emerged, notably **Alexandria, Egypt**, with its famous Library and Museum
- Mathematics became more specialised, moving beyond fundamental geometry into specific applications and new areas
- Focus often shifted from pure, abstract geometry to areas with practical utility (astronomy, optics, mechanics)

Mathematics in the Hellenistic era

Key figures and fields:

- *Apollonius of Perga*: Conic sections
- *Hipparchus & Ptolemy*: Astronomy and trigonometry
- *Eratosthenes*: Geography and Earth measurement
- *Diophantus of Alexandria*: Proto-algebra and number theory
- *Pappus of Alexandria*: Compiler and commentator

A period of consolidation and innovation: While pure geometry reached its zenith with Euclid and Archimedes, the Hellenistic period saw the application of these powerful tools to complex problems, and the beginnings of algebraic thought.

Apollonius of Perga

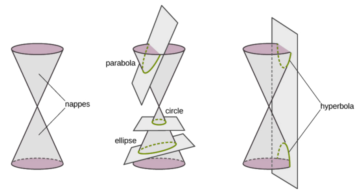
Apollonius of Perga (~240–190 BCE):

- Often called "The Great Geometer".
- Student of the Alexandrian school
- His magnum opus: "**Conics**" (8 books, 7 survive)

Conic sections

The conic sections:

- *Parabola*, *ellipse*, *hyperbola*
- Apollonius showed that these curves arise from cutting a **single cone** (double-napped) at different angles, unlike earlier methods that used different cones



Conic sections from a double-napped cone

- He introduced the terms 'parabola', 'ellipse', and 'hyperbola'
- Studied their properties comprehensively, developing the theory in a way that approaches analytical geometry without coordinates

Later importance: Apollonius's work on conics became fundamental for Kepler's laws of planetary motion and Newton's law of universal gravitation, centuries later

Astronomy, trigonometry

Trigonometry for Astronomy:

- Greek astronomers needed to calculate distances and angles in the celestial sphere
- *Hipparchus of Nicaea* (~190–120 BCE):
  - Considered the "father of trigonometry".
  - Created the first known table of chords for various angles (equivalent to sine values: $2\sin(\alpha) = \text{chord}(2\alpha)$)
  - Used these to construct precise astronomical models
- *Ptolemy* (~100–170 CE):
  - His "Almagest" (mathematical treatise on astronomy) codified Greek trigonometry and astronomical models (geocentric)
  - His chord table extended Hipparchus's work

Global measurements

Eratosthenes of Cyrene (~276–195 BCE):

- Chief librarian at the Library of Alexandria
- Pioneer in geography and cartography
- Calculated the **circumference of the Earth** with remarkable accuracy

Eratosthenes' method (simplified):

- Noted that at noon on the summer solstice, sunlight directly illuminated a deep well in Syene (modern Aswan)
- At the same time in Alexandria, a vertical stick cast a shadow, indicating the sun was 7.2 degrees from vertical
- Assuming the sun's rays are parallel and Earth is a sphere, the angle 7.2° represents the angle between Syene and Alexandria at the Earth's centre
- Calculated the distance between Syene and Alexandria (~ 5000 stadia)
- Circumference $= (360°/7.2°) \times 5000$ stadia $= 50 \times 5000 = 250{,}000$ stadia
- This means ~39,690 km (actual ~40,075 km)—astounding precision!

Diophantus of Alexandria

Diophantus of Alexandria (~200–284 CE):

- A Hellenistic mathematician, much later than the classical geometers
- His main work: "**Arithmetica**" (13 books, 6 survive in Greek, 4 more in Arabic)
- Diverged significantly from the traditional Greek geometric approach

Focus of Arithmetica:

- Solves hundreds of specific problems involving rational solutions to indeterminate equations (equations with more variables than equations)
- These are now known as **Diophantine equations**
- Example: Find three numbers such that the product of any two added to the third is a perfect square

The dawn of algebra

Key innovations: Proto-symbolic algebra

- While not fully symbolic like modern algebra, Diophantus used abbreviations and a kind of syncopated algebra
- Example: He would use **s** for the unknown (x), **ds** for x^2, **cs** for x^3, etc.
- This was a radical departure from the purely rhetorical (word-based) algebra of earlier periods and the geometric algebra of Euclid

Significance:

- Bridges the gap between Babylonian numerical problems and later Arabic/European algebra
- Showed that abstract numerical problems could be solved systematically, not just geometric ones
- His work was later influential for mathematicians like Fermat

Pappus of Alexandria

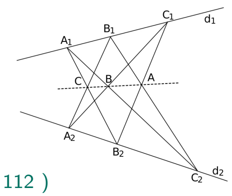
Pappus of Alexandria (~290–350 CE):

- One of the last great Greek mathematicians of antiquity
- Most famous work: "**Collection**" (Synagoge), an 8-book summary of Greek mathematics
- Preserved, summarised, and commented on a vast amount of earlier Greek mathematical knowledge that would otherwise be lost
- Also contained some original theorems and problems (e.g., Pappus's hexagon theorem, Pappus's centroid theorems)

Pappus of Alexandria

Centroid theorems:

- The area A of a surface generated by rotating a plane curve C about an axis external to C and on the same plane is $A = sd$ with s the arc length of C and d the distance travelled by the geometric centroid of C
- The volume V of a solid generated by rotating a plane figure F about an external axis is $V = Ad$ with A the area of F and d the distance travelled by the geometric centroid of F
- For example, the surface area of the torus with minor radius r and major radius R is $A = (2\pi r)(2\pi R) = 4\pi^2 Rr$ and the volume is $V = (\pi r^2)(2\pi R) = 2\pi^2 Rr^2$



Hexagon theorem: Fixing the points A_1, B_1 and C_1 on the line d_1, and A_2, B_2 and C_2 on the line d_2 as in the figure aside, the intersections A, B, C are on the same line

The end of an era

The decline of original Greek mathematics:

- After Pappus (and Hypatia), the pace of original mathematical contribution slowed significantly
- **Factors contributing to decline**:
  - Roman Empire's focus on practical administration and engineering, not abstract pure science
  - Political and economic instability (e.g., Sack of Alexandria)
  - Rise of new religious ideologies (Christianity) that sometimes clashed with pagan philosophical inquiry
  - Destruction of the Library of Alexandria over centuries
  - Loss of institutional support and patronage for pure research
- Mathematics largely preserved and developed by **Islamic scholars** in the centuries that followed

The end of an era

A legacy that endures: Despite the decline, the rigorous, deductive framework established by the Greeks, especially by Euclid, Archimedes, and Apollonius, laid the essential groundwork for all future mathematical advancements, particularly in Europe during the Renaissance

The enduring legacy of Greek mathematics

A paradigm shift that shaped all science

- **The birth of deduction and proof**:
  - Transition from "how" (recipes, algorithms) to "why" (rigorous, logical argument)
  - Mathematics established as a science based on axioms, definitions, and theorems
- **The axiomatic method** (Euclid):
  - A revolutionary framework for organising knowledge, serving as a model for all future scientific inquiry
  - Demonstrated how complex structures can be built from simple, self-evident truths
- **Confronting the infinite**:
  - Zeno's paradoxes highlighted foundational challenges
  - Eudoxus's theory of proportion and method of exhaustion provided rigorous ways to handle continuous magnitudes and limit-like processes

The enduring legacy of Greek mathematics

A paradigm shift that shaped all science

- **Pinnacle of geometric genius** (Archimedes, Apollonius):
  - Masterful treatment of areas, volumes, and conic sections, foreshadowing integral calculus
  - Unparalleled precision (e.g., π, Earth's circumference)
- **Foundations for future fields**:
  - Early number theory (Euclid, Diophantus)
  - Trigonometry (Hipparchus, Ptolemy) and its astronomical applications
  - Proto-algebra (Diophantus)

Greek mathematics established the intellectual toolkit and the standard of rigour that would define mathematics for over two millennia, influencing the scientific revolution and modern thought

References

The lesson is based on:

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Dirk J. Struik* A Concise History of Mathematics, Fourth revised edition, Dover Publications (1987)

Apollonius's Conics, *Diophantus*' Arithmetica, and *Ptolemy*'s Almagest are freely available online in a reasonable English translation. *Pappus*' Synagoge doesn't have a full English translation yet: a partial Latin version and Book VIII can be found online.

For the reader's enjoyment *Denis Guedj*, La chioma di Berenice, Longanesi (2003) is a novel about Eratosthenes's measure of the world

History of Mathematics: Lecture 5

Syllabus:

Ancient Asian Mathematics:

- The revolutionary contributions of Indian mathematics
- The practical and algorithmic power of Chinese mathematics
- The unique geometric aesthetics of Japanese (Wasan) mathematics

Indian mathematics: A conceptual revolution

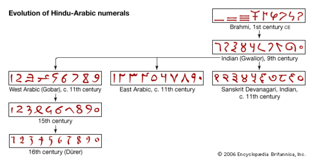
Context and periods:

- Rich mathematical tradition spanning millennia (from Vedic period onwards)
- Influenced by and influenced astronomy and religious texts
- Key periods and figures:
  - **Gupta period** (~320–550 CE): *Aryabhata*
  - **Classical period** (~6^{th}–12^{th} CE): *Brahmagupta*, *Bhaskara II*
  - **Kerala school** (~14^{th}–16^{th} CE): *Madhava*

Indian mathematics: A conceptual revolution

The greatest contribution:
The decimal place-value system

- Developed by Indian mathematicians, likely between 1^{st} and 5^{th} centuries CE
- **Key features**:
  - Nine distinct digits (1-9)
  - **Positional value**: The value of a digit depends on its position (e.g., '2' in 200 vs. 20)
  - The conceptual innovation of **zero as a placeholder and a number** (*Shunya*)
- This system vastly simplified arithmetic, making complex calculations manageable



Global impact: This system, later adopted by Arab scholars and then transmitted to Europe, is the **foundation of modern arithmetic worldwide**

Algebra and negative numbers

Early concept of negative numbers:

- Appeared as early as **Brahmagupta** (~598–668 CE) in his *Brahmasphutasiddhanta*
- Treated negative numbers as "debts" or "losses", acknowledging their existence and rules for operations:
  - A debt minus a debt is a debt
  - A fortune minus a debt is a fortune
  - The product of two debts is a fortune, i.e., $(-x)(-y) = xy$
- This predates the full acceptance of negative numbers in Europe by centuries

Solving quadratic equations:

- Brahmagupta gave the general solution to quadratic equations in the form $ax^2 + bx = c$
- He allowed for two roots, including negative ones (though he sometimes discarded negative roots in practical contexts)

Indeterminate equations

Indeterminate or Diophantine equations:

- Indian mathematicians extensively studied equations with integer solutions (linear and quadratic)
- **Brahmagupta** solved linear indeterminate equations of the form $ax + by = c$
- **Pell's equation**: The equation $Nx^2 + 1 = y^2$ (where N is an integer)
  - Methods to solve this were developed by Brahmagupta (the "Brahmagupta identity") and later extended by **Bhaskara II** (~1114–1185 CE) using the "Chakravala" (cyclic) method
  - This was a highly sophisticated achievement, far more advanced than anything in Europe until the 17^{th}–18^{th} centuries (the name of Pell's equation arose from Leonhard Euler mistakenly attributing Brouncker's solution of the equation to John Pell)

Emphasis on algorithms: Indian mathematics often presented solutions as clear algorithms or rules, similar to Mesopotamian mathematics, but with greater generality and abstraction

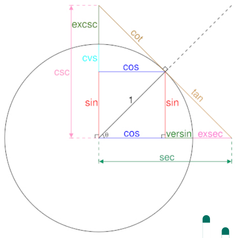
Trigonometry

Development of trigonometry (for astronomy):

- Unlike the Greek use of chords, Indian mathematicians developed the concepts of **sine** and **cosine**
- **Aryabhata** (~476–550 CE) in his *Aryabhatiya*:
  - Created the first sine table (*Jya*) and versine table (*Koti-jya*) at intervals of 3.75°
  - His methods for calculating these values were ingenious
- These concepts were later transmitted to the Islamic world and then to Europe

Mathematical astronomy:

- Indian mathematicians were also accomplished astronomers, developing sophisticated models and calculations
- This practical application heavily influenced their mathematical developments



The dawn of infinite series

The Kerala School of Mathematics (~14^{th}–16^{th} CE):

- A remarkable school in South India (Kerala region)
- Figures like **Madhava of Sangamagrama** (~1340–1425 CE)
- Developed infinite series expansions for trigonometric functions:
  - Madhava-Leibniz series for $\pi/4$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

  - Madhava-Gregory series for $\arctan(x)$:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

  - Madhava's series for sine and cosine (Taylor series expansions)
- These discoveries predate European work by more than two centuries!
- They also worked on geometric series and early ideas of differentiation

The dawn of infinite series

A proto-calculus? The Kerala School's work came remarkably close to developing calculus, though it remained largely localised for centuries

Chinese mathematics

Context and characteristics:

- Long and continuous tradition, driven by needs of bureaucracy, engineering, and astronomy
- Emphasis on **algorithms and methods for computation** rather than formal deductive proofs (differing from Greeks)
- Utilised a sophisticated **decimal place-value system** (using counting rods) very early on

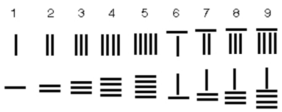
Major periods and texts:

- **Han Dynasty** (~206 BCE–220 CE): *The Nine Chapters on the Mathematical Art*
- **Three Kingdoms/Tang Dynasty** (~3rd–10th CE): Liu Hui, Zu Chongzhi
- **Song/Yuan Dynasty** (~10th–14th CE): Jia Xian, Yang Hui, Qin Jiushao, Li Ye, Zhu Shijie. Peak of Chinese algebra

Chinese mathematics

Key tool: *Counting rods* (筹算, chou suan):

- Wooden or bamboo rods used for calculation, arranged in columns for powers of 10
- Allowed for representation of positive and negative numbers, and even zero (represented by a blank space)
- Facilitated efficient arithmetic operations and early forms of matrix manipulation



Chinese counting rods (early place-value system)

The Nine Chapters and linear systems

Jiǔzhāng Suànshù (九章算術): *The Nine Chapters on the Mathematical Art*

- Compiled likely in the **Han Dynasty** (~1st Century CE), with later commentaries (Liu Hui, 3rd CE)
- A collection of **246 problems** divided into nine chapters, covering practical applications
- Each problem provides a statement, an answer, and a detailed algorithm for solving it
- Wide range of topics: surveying, engineering, taxation, agriculture, commerce, geometry

The Nine Chapters and linear systems

Solving systems of linear equations:

- Chapter 8, "Fangcheng" (方程[1], "Rectangular Arrays" or "The Way of Calculation")
- Describes a method for solving systems of linear equations with multiple unknowns
- This method is strikingly similar to modern **Gaussian elimination** or matrix methods
- Utilised counting rods arranged on a counting board to represent coefficients in a rectangular array

[1] In modern Chinese it translates to "equation"

The Nine Chapters and linear systems

Example problem (from Chapter 8):

- "There are 3 types of corn: 3 bundles of the first type, 2 of the second, 1 of the third, sum to 39 dou of grain. 2 of the first, 3 of the second, 1 of the third, sum to 34 dou. 1 of the first, 2 of the second, 3 of the third, sum to 26 dou. Find the amount of grain in one bundle of each type."
- Corresponds to the system:

$$3x + 2y + z = 39$$
$$2x + 3y + z = 34$$
$$x + 2y + 3z = 26$$

- The Chinese method involved systematically manipulating the columns of the rod array to eliminate variables

Advanced for its time: This method predates its independent rediscovery in the West by many centuries, showcasing remarkable algebraic intuition

Geometry and π approximations

- Chapter 1 of "Nine Chapters" deals with areas
- **Liu Hui** (~3rd century CE, commentator on "Nine Chapters")
  - Used the method of inscribed polygons to approximate π
  - Calculated $\pi \approx 3.14159$
  - Described his "circle area algorithm" (割圆术, "cutting the circle method") similar to Archimedes'
- **Zu Chongzhi** (~5th century CE):
  - Calculated π to be between 3.1415926 and 3.1415927
  - Gave the famous fractions 22/7 (milü, "rough ratio") and 355/113 (miqie, "close ratio").
  - 355/113 is remarkably accurate (correct to 6 decimal places), unsurpassed for nearly 1000 years

Advanced algebra

Higher-degree equations (Song/Yuan dynasty):

- Chinese mathematicians developed numerical methods to find roots of polynomial equations of degrees up to 10
- **Qin Jiushao** (~13th century CE): Applied a method similar to **Ruffini-Horner's method** (known in China as the *tian yuan shu*, "celestial element method" or "horn method")
- This method could find numerical approximations for roots to any desired degree of accuracy

Advanced algebra

The Ruffini-Horner method provides a fast way to evaluate a polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ in a point x_0.
For this, a new sequence of constants is recursively defined as:

$$b_n = a_n$$
$$b_i = a_i + b_{i+1} x_0$$

for $0 \le i < n$. Then $b_0 = p(x_0)$

Similarly, it can be shown that

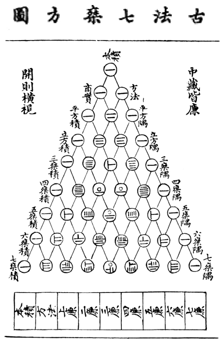
$$p(x) = \left(b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \cdots + b_{n-1} x^{n-2} + b_n x^{n-1}\right)(x - x_0) + b_0$$

suggesting a convenient procedure for determining the result of the polynomial division $p(x)/(x - x_0)$

Advanced algebra

Pascal's (Yang Hui's) Triangle:

- For binomial coefficients of $(a+b)^n$
- Depicted by **Jia Xian** (~11th century CE) and then **Yang Hui** (~13th century CE), centuries before Pascal.
- Used for solving polynomial equations



Yang Hui's Triangle

A diverse and deep tradition: Chinese mathematics developed a unique and powerful algorithmic approach that yielded impressive results in numerical methods and algebra

Japanese mathematics (Wasan)

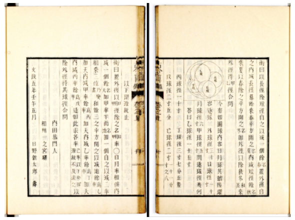
Wasan (和算): Japanese Mathematics

- Developed in Japan, distinct from Western mathematics
- Heavily influenced by **Chinese mathematics** (via Korea), especially in its early stages
- Flourished primarily during the **Edo period** (1603–1868), a time of national isolation (Sakoku policy)
- This isolation allowed Wasan to develop along its own unique trajectory, focusing on specific types of problems and methodologies

Japanese mathematics (Wasan)

Characteristics of Wasan:

- Emphasis on **geometric problems**, often very intricate
- Solutions usually presented as algorithms, often without formal proofs in contrast with Greek tradition
- Often highly competitive, with mathematicians challenging each other



Kokon sankan, Mathematical Library, Department of Mathematics, Kyoto University

Key figures:

- **Seki Takakazu** (~1642–1708): Often called the "Newton of Japan"
- **Takebe Katahiro** (~1664–1739): Pupil of Seki

Sangaku

Sangaku (算額): Temple Geometry

- Beautiful wooden tablets hung in Shinto shrines and Buddhist temples
- Contained intricate geometric problems and their solutions (sometimes just the problem statement as a challenge)
- Offered by individuals to deities as offerings, or as intellectual challenges
- Showcases a unique blend of mathematics, art, and spirituality
- Problems often involved circles, ellipses, and other curves tangent to each other within larger shapes

Examples of Sangaku problems

Seki Takakazu

Seki Takakazu's advanced contributions:

- Developed a new algebraic notation system
- Pioneered the concept of **determinants** (around 1683), used for solving systems of linear equations, predating Leibniz's work
- Discovered **Bernoulli numbers** independently (used in sums of powers)
- Developed methods for calculating definite integrals and volumes, including early ideas akin to **calculus**, (he introduced a form of infinitesimal calculus using "enri" or circle principle)
- Approximated π up to ten digits; his school arrived up to 18
- His work remained largely unknown outside Japan due to isolation

A path diverted: Wasan showcased a highly sophisticated mathematical culture, but its isolation meant its innovations did not directly influence the global mathematical mainstream until much later

Bernoulli numbers

Bernoulli numbers are used to compute a closed form for $\sum_{i=i}^{n} i^m$ with m a fixed positive integer:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\sum_{i=0}^{m}\binom{m+1}{i}B_i n^{m-i+1}$$

The first Bernoulli numbers are $B_0 = 1$, $B_1 = 1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$. There is a recursive formula to compute all of them

If $m = 1$ then

$$\sum_{i=1}^{n} i = \frac{1}{2}\left(B_0 n^2 + 2B_1 n\right) = \frac{1}{2}\left(n^2 + n\right)$$

If $m = 4$ then

$$\sum_{i=1}^{n} i^4 = \frac{1}{5}\sum_{j=0}^{4}\binom{5}{j}B_j n^{5-j} = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

Global contributions to early mathematics

Diverse paths, universal principles

- **Indian Mathematics**:
  - The revolutionary **decimal place-value system** and the concept of **zero**
  - Advanced algebra (negative numbers, Pell's equation)
  - Origins of sine/cosine functions and the remarkable **Kerala School's infinite series** (proto-calculus)
- **Chinese Mathematics**:
  - Highly practical and algorithmic approach.
  - Sophisticated methods for solving **linear systems** (Gaussian elimination)
  - Accurate π approximations and solutions for **higher-degree equations**
  - Independent discovery of **Pascal's triangle**
- **Japanese (Wasan) Mathematics**:
  - Unique development during isolation, often in the form of elegant geometric puzzles (**Sangaku**).
  - Advanced concepts like **determinants** and early calculus methods (*enri*) by Seki Takakazu

Global contributions to early mathematics

These rich and diverse traditions demonstrate that fundamental mathematical ideas emerged independently across different cultures, often driven by unique intellectual and practical needs, contributing immensely to the global tapestry of mathematical knowledge

References

The lesson is based on:

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)

While it is not difficult to find resources on Indian and Chinese Mathematics, access to the original sources or to reasonable translations is difficult

A beautiful book about Japanese Mathematics is *Fukagawa Hidetoshi*, *Tony Rothman*, Sacred Mathematics: Japanese Temple Geometry, Princeton University Press (2008)

History of Mathematics: Lecture 6

Syllabus:

The Islamic Golden Age

- A bridge of knowledge and innovation

The Islamic Golden Age

A period of unprecedented intellectual flourishing:

- Spanning roughly the **8$^{\text{th}}$** to the **14$^{\text{th}}$** centuries CE
- Characterised by the collection, translation, and vigorous advancement of knowledge from diverse cultures
- Mathematics was a central pillar, fuelled by religious (e.g., qibla, inheritance), administrative (e.g., taxation, land division), and scientific (e.g., astronomy) needs

The historical context

The rise of the Islamic empire:

- Rapid expansion from the 7th century CE, connecting vast regions from Spain to India
- This created a unique cultural melting pot, bringing together Persian, Indian, Greek, and other intellectual traditions

The Abbasid caliphate

The Abbasid Caliphate (~750–1258 CE):

- Shifted political power eastward from Damascus to Baghdad
- Patronised learning, science, and the arts as a matter of state policy
- Sought to consolidate knowledge from across their vast empire and beyond

The Abbasid Caliphate at its height

Baghdad

Baghdad: The intellectual capital

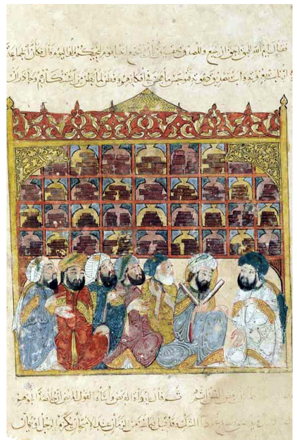
- Founded in 762 CE, it quickly became the largest city in the world and a centre of commerce and learning
- Home to scholars from diverse backgrounds and faiths, fostering an environment of open inquiry

A civilisation built on knowledge: The political stability and active patronage created fertile ground for intellectual pursuits, particularly in mathematics and astronomy

The House of Wisdom

The House of Wisdom (Bayt al-Hikmah):

- Established in Baghdad by Caliph Harun al-Rashid, expanded significantly under Caliph al-Ma'mun (~early 9th century CE)
- More than just a library: it was an academy, translation bureau, and research centre
- Attracted scholars (mathematicians, astronomers, physicians, philosophers) from across the empire and beyond



The House of Wisdom

The translation movement

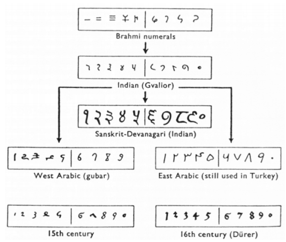
- A massive, systematic effort to translate ancient texts into Arabic
- **Greek texts**: Euclid's "Elements", Ptolemy's "Almagest", Archimedes' works, Diophantus's "Arithmetica"
- **Indian texts**: Siddhantas (astronomical treatises), mathematical works on decimal system and algebra
- **Persian texts**; Astronomical tables and other scientific works
- Funded lavishly; translators were highly respected and well-paid

More than just translation: Scholars didn't just translate; they **commented**, **critiqued**, **synthesised**, and **expanded** upon the ancient knowledge, often correcting errors and resolving inconsistencies. This was the start of original research

The Hindu-Arabic numeral system

The power of a symbol system:

- Recall the Indian invention of the **decimal place-value system** and the concept of **zero**
- Islamic scholars recognised its immense superiority over existing numeral systems (e.g., Roman numerals, Greek alphabetic numerals)
- Allowed for vastly more efficient arithmetic operations (addition, subtraction, multiplication, division)



Evolution of Hindu-Arabic Numerals (via Islamic scholarship)

The Hindu-Arabic numeral system

Adoption and transmission:

- Introduced to the Islamic world primarily through translations of Indian astronomical and mathematical texts (e.g., Brahmagupta's works)
- **Al-Khwarizmi's** text, *Book on Addition and Subtraction after the Method of the Hindus* (~825 CE), was crucial
- This system was then transmitted to Europe through trade, scholarship, and conquest (e.g., through Islamic Spain, Sicily)
- It took centuries for Europe to fully adopt it over Roman numerals

Impact on calculation:

- Made long multiplication and division practical
- Facilitated the development of commercial arithmetic and advanced astronomical calculations
- Paved the way for decimal fractions

Muhammad ibn Musa al-Khwarizmi

A scholar of the House of Wisdom (~780–850 CE):

- Persian mathematician and astronomer, active in the House of Wisdom in Baghdad
- His two most influential works profoundly shaped global mathematics:
  1. *Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa'l-muqābala*
     (The Compendious Book on Calculation by Completion and Balancing)
  2. *Kitāb al-Jam wa'l-tafrīq bi-ḥisāb al-Hind*
     (Book on Addition and Subtraction after the Method of the Hindus)

Muhammad ibn Musa al-Khwarizmi

The origin of algebra:

- The term "**Algebra**" comes from the Arabic title of his first book, ***al-jabr***
- *Al-jabr*: "completion" or "restoration", referring to transposing negative terms to the other side of an equation to make them positive, e.g., $x-2=5 \implies x=5+2$
- *Al-muqābala*: "balancing" or "reduction", referring to combining like terms on opposite sides of an equation, e.g., $2x+5=x+7 \implies x+5=7$

Muhammad ibn Musa al-Khwarizmi

The origin of algorithms:

- The term "**Algorithm**" derives from the Latinisation of Al-Khwarizmi's name: "**Algorismi**"
- It refers to any systematic procedure for solving a mathematical problem
- His second book, describing arithmetic with Indian numerals, was foundational for teaching these "algorithms"

A new mathematical discipline: Al-Khwarizmi defined **algebra** as a distinct mathematical discipline, systematically solving linear and quadratic equations, moving beyond specific problems to general methods

Systematic approach to quadratic equations

Classification of quadratics:

- Al-Khwarizmi recognised three canonical forms of quadratic equations (all terms positive, as negative numbers were not yet fully accepted in this context for solutions):
  1. "Squares equal to roots": $ax^2=bx$, e.g., $x^2=5x$
  2. "Squares equal to numbers": $ax^2=c$, e.g., $x^2=9$
  3. "Roots equal to numbers": $bx=c$. e.g., $3x=12$
- And three mixed cases:
  4. "Squares and roots equal to numbers": $ax^2+bx=c$, e.g., $x^2+10x=39$
  5. "Squares and numbers equal to roots": $ax^2+c=bx$, e.g., $x^2+21=10x$
  6. "Roots and numbers equal to squares": $bx+c=ax^2$, e.g., $3x+4=x^2$

Systematic approach to quadratic equations

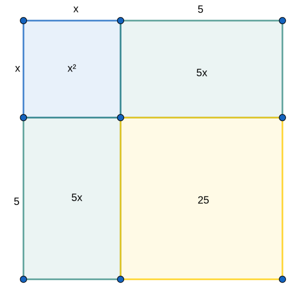
Solution Method (Example: $x^2+10x=39$):

- His methods were described in words (rhetorical algebra), not symbols
- He provided **geometric proofs** for the solutions
- For $x^2+10x=39$:
  1. "Halve the number of roots, which is 5. Multiply this by itself, 25. Add this to 39, which is 64. Take the root, 8. Subtract half the number of roots, 5. The result is 3"
  2. Thus, $x=3$. (Equivalent to $x=\sqrt{(10/2)^2+39}-(10/2)$)

Systematic approach to quadratic equations

Geometric justification
(completing the square):

- Represent x^2 as a square, $10x$ as two rectangles ($5x$ each)
- "Complete the square" by adding a square of area $5^2 = 25$
- The total area is $x^2 + 10x + 25 = (x+5)^2 = 39 + 25 = 64$
- So, $x + 5 = \sqrt{64} = 8 \implies x = 3$



Al-Khwarizmi's geometric method for $x^2 + 10x = 39$

Advanced algebra: Beyond the quadratic

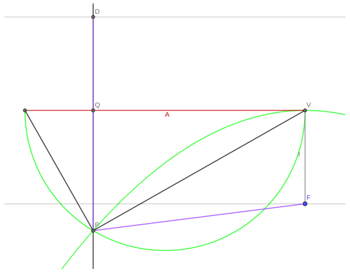
Omar Khayyam (~1048–1131 CE):

- Persian polymath: mathematician, astronomer, philosopher, and poet
- Wrote *Treatise on Demonstration of Problems of Algebra*, classifying and solving **cubic equations**
- His method was primarily **geometric**: finding the roots as the intersection points of conic sections (parabolas, hyperbolas, circles)
- Recognised that cubics could have multiple roots and sometimes no positive

Advanced algebra: Beyond the quadratic

Example: Solving $x^3 + bx = a$ geometrically

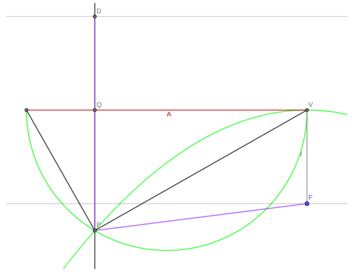
- First, the equation is rewritten as $x^3 + B^2x = B^2A$ where $B = \sqrt{b}$ and $A = a/b$. Let $f = B/4$
- Khayyam would find the intersection of a parabola of focus F and vertex V with $\overline{FV} = f$, and the semi-circle of diameter A as in figure
- The intersection P determines the length $r = \overline{QV}$, which is a root, and the height $z = \overline{PQ}$



Advanced algebra: Beyond the quadratic

Example: Solving $x^3 + bx = a$ geometrically

- Since P lies on a parabola, $\overline{PF} = \overline{PD} = z + f$
- But $r^2 + (z-f)^2 = \overline{PF}^2$ by Pythagoras' Theorem, so $r^2 = 4zf$, that is, $z = r^2/B$
- Also, $\frac{A-r}{z} = \frac{z}{r}$ because P and the diameter form a right triangle
- Thus $B(A-r)/r^2 = r/B$, that is, $r^3 + B^2r = B^2A$, so r is indeed a root



This linked algebra to geometry in a powerful way, extending methods from the Greeks.

Advanced algebra: Beyond the quadratic

Sharaf al-Dīn al-Tūsī (~1135–1213 CE):

- Persian mathematician
- Developed a truly **algebraic method** for solving cubic equations that involved finding the maximum value of a cubic polynomial
- This implied an early understanding of **derivatives** and polynomial roots, a significant step towards algebraic calculus

Bridging geometry and algebra: These methods pushed the boundaries of algebra by employing both geometric visualisation and new analytical techniques to solve problems previously intractable

Number theory and combinatorics

Thabit ibn Qurra (~826–901 CE): Amicable numbers

- A prominent scholar of the House of Wisdom
- Made significant contributions to number theory, geometry, and astronomy
- Discovered a theorem for finding **amicable numbers**
- **Amicable numbers** Two different numbers such that the sum of the proper divisors of each equals the other
  - Example: 220 and 284
  - Proper divisors of 220: 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110. Sum = 284
  - Proper divisors of 284: 1, 2, 4, 71, 142. Sum = 220
- Thabit's formula: If $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$, $r = 9 \cdot 2^{2n-1} - 1$ are all prime numbers for $n > 1$, then $2^n pq$ and $2^n r$ are amicable numbers

Number theory and combinatorics

Combinatorics: Binomial coefficients and Pascal's triangle

- Recall the "Yang Hui's triangle" from Chinese mathematics
- Islamic mathematicians also studied arrangements and combinations, essential for cryptography, grammar, and astronomy
- **Al-Karaji** (~953–1029 CE): Extended algebraic operations to powers and roots, using a form of the binomial theorem for integer exponents
- **Al-Samaw'al** (~1130–1180 CE): Further developed work on polynomial division and negative powers
- **Al-Kashi** (~1380–1429 CE): Explicitly presented a version of **Pascal's triangle** for calculating binomial coefficients, long before Pascal

From number patterns to general rules: These contributions demonstrate a sophisticated understanding of number theory and combinatorial principles that anticipated later European developments

The full development of trigonometry

Beyond chords: From India to a complete system

- Recall Greek trigonometry (chords) and Indian trigonometry (sine/cosine)
- Islamic scholars translated and absorbed both traditions, then significantly expanded them
- Driven by the needs of astronomy (celestial navigation, calculating qibla direction), they systematised and generalised trigonometric concepts

Key figures and innovations:

- **Al-Battani** (~858–929 CE, "Albategnius"):
  - Introduced the use of the **sine and tangent functions** systematically
  - Created the first tables for all sine and tangent values from 0° to 90° at 1° intervals
  - Used trigonometric ratios in his astronomical calculations
- **Abu al-Wafa' al-Buzjani** (~940–998 CE):
  - Introduced the **secant and cosecant functions**
  - Developed improved methods for constructing sine tables

The full development of trigonometry

Modern trigonometry emerges:

- Islamic mathematicians had defined **all six trigonometric functions**: sine, cosine, tangent, cotangent, secant, cosecant
- They established the fundamental **relationships between these functions** (e.g., $\tan x = \sin x / \cos x$)
- Developed highly accurate trigonometric tables

A foundation for modern calculation:
The complete system of trigonometry developed in the Islamic world by the 10^{th} century was a direct precursor to the European rebirth of the field and enabled precise calculations in navigation and astronomy

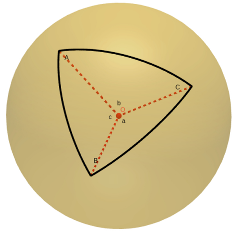
The full development of trigonometry

Spherical trigonometry:

- Crucial for astronomy and calculating directions on the Earth's surface (e.g., direction to Mecca, the **qibla**)
- Developed the **law of sines** and the laws of cosines for spherical triangles

$$\frac{\sin(a)}{\sin(A)} = \frac{\sin(b)}{\sin(B)} = \frac{\sin(c)}{\sin(C)}$$

- **Nasir al-Din al-Tusi** (~1201–1274 CE): Wrote the first treatise on trigonometry independent of astronomy



Constructions and insights into Euclid

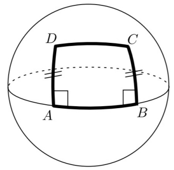
Refining Euclidean geometry:

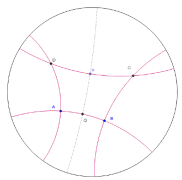
- Islamic scholars rigorously studied, translated, and commented on Euclid's "Elements"
- Many mathematicians attempted to "prove" Euclid's controversial **Parallel postulate**

The parallel postulate and its implications:

- "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles"
- Attempts to prove it led to defining what we now call **Saccheri quadrilaterals** and exploring alternatives

Constructions and insights into Euclid





A *Saccheri quadrilateral ABCD* has the *legs AD* and *BC* are equal in length and each perpendicular to the base *AB*; the top *CD* is called the *summit*

The existence of a Saccheri quadrilateral with right angles *C* and *D* for any base and sides is equivalent to the parallel postulate, leading to Euclidean geometry. In hyperbolic geometry, arising from the negation of the parallel postulate, *C* and *D* angles are always acute. In elliptic or spherical geometry (which require a few modifications to Euclid's other postulates) the *C* and *D* are always obtuse

Constructions and insights into Euclid

- **Omar Khayyam** (11^{th} CE) and **Nasir al-Din al-Tusi** (13^{th} CE) constructed various equivalent statements and explored the consequences of denying it
- Their work laid crucial groundwork for the later development of **Non-Euclidean geometries** in the 19^{th} century

Constructions and insights into Euclid

Islamic geometric patterns: mathematics as art

Geometric constructions and transformations:

- Developed sophisticated geometric constructions, often using straightedge and compass
- Investigated geometric transformations (rotations, reflections, translations) in relation to art and architecture (e.g., tessellations, patterns in mosques)
- **Ibn al-Haytham** (~965–1040 CE, "Alhazen"):
  - Pioneer in optics, solving "Alhazen's problem" (finding the point on a spherical mirror where light reflects to a given point). This involved solving a fourth-degree equation
  - His geometrical work on optics had profound influence
- Islamic geometry was not only theoretical but also deeply integrated into art, architecture, and practical engineering

Astronomy

Astronomy as a scientific imperative:

- Religious obligations (prayer times, qibla direction, lunar calendar) provided strong motivation for accurate astronomy
- Navigational needs for trade and pilgrimage
- Islamic astronomers built upon Greek (Ptolemy) and Indian (Siddhantas) models, making extensive new observations

New observatories and instruments:

- Established sophisticated observatories (e.g., Maragha, Samarkand), funded by rulers
- Developed highly accurate instruments:
  - **Astrolabe**: Used for solving astronomical problems (e.g., finding time, latitude, positions of stars)
  - **Quadrant and sextant**: For measuring angles of celestial bodies
- These required advanced mathematical understanding for their design and use

Astronomy

Refining astronomical models:

- Critiqued and improved Ptolemaic models, sometimes proposing alternatives to geocentric models
- **Al-Biruni** (~973–1048 CE):
  - Polymath, made significant contributions to geography (measured Earth's radius accurately), astronomy, and mapping
  - Discussed Earth's rotation
- **Al-Tusi's couple:** A geometric construction that explained the linear motion from two circular motions, used to replace the equant in Ptolemaic models

An intricately crafted Islamic astrolabe

Mathematics as the language of the cosmos: Mathematical tools were indispensable for unravelling the mysteries of the universe, leading to increasingly precise understanding of celestial mechanics

A global catalyst for mathematics

From preservation to illumination and transmission

- **Guardians of ancient knowledge**:
  - Preserved, translated, and critically engaged with Greek (Euclid, Ptolemy, Archimedes) and Indian (decimal system, algebra) mathematical texts
  - Acted as the vital intellectual bridge between antiquity and the European Renaissance
- **Revolutionary innovations**:
  - **Algebra** established as a distinct field (Al-Khwarizmi)
  - Widespread adoption and spread of the **Hindu-Arabic numeral system and zero**
  - Development of the **full system of trigonometry** (all six functions, spherical trigonometry)
  - Advances in number theory (amicable numbers, Thabit ibn Qurra) and methods for solving **higher-degree equations** (Omar Khayyam, Sharaf al-Din al-Tusi)
  - Proto-concepts of **non-Euclidean geometry** and **calculus**

A global catalyst for mathematics

From preservation to illumination and transmission

- **Interdisciplinary excellence**:
  - Deep integration of mathematics with astronomy, engineering, optics, and architecture
  - Development of advanced scientific instruments and observatories

The mathematical achievements of the Islamic Golden Age were not merely a re-telling of past ideas, but a dynamic crucible of original thought, refinement, and transmission that fundamentally shaped the course of global scientific progress for centuries

References

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History of Mathematics: Lecture 7

UNIVERSITAS STUDIORUM INSUBRIAE

Syllabus:

- The Middle Ages
- The Renaissance

European mathematics: The middle ages

Beyond the "dark ages" myth: preservation and practicality

What was the European middle ages?

- Period between the fall of the Western Roman empire and the Renaissance (~500–1400 CE)
- Characterised by localised political structures, the rise of Christianity, and evolving social systems

Mathematics in the "dark ages"?

- The term "dark ages" is largely a misnomer, especially for intellectual life
- **Original mathematical innovation was limited**, particularly compared to the Islamic world or ancient Greece/India
- Focus was on **preservation**, basic utility, and theological interpretation of classical knowledge

Medieval mathematical endeavours

Centres of learning:

- **Monasteries and Cathedrals**: Key repositories of knowledge, copying manuscripts (including classical texts)
- **Early universities (11^{th}–12^{th} CE onwards)**: Formed in Bologna, Paris, Oxford; initially focused on theology, law, medicine, and the liberal arts
- Mathematics taught as part of the **Quadrivium** (arithmetic, geometry, astronomy, music) within the seven liberal arts

Medieval Scriptorium: Preserving knowledge through copying

Medieval mathematical endeavours

University of Bologna

Key practical applications:

- **Computus:** The complex calculation of the date of Easter, requiring astronomical and arithmetical knowledge
- **Rudimentary arithmetic:** For commerce, accounting, and basic surveying
- **The abacus:** Remained the primary computational tool, often using Roman numerals

Limited innovation, vital preservation: While not a period of dramatic mathematical breakthroughs in Europe, the Middle Ages were crucial for preserving a fragmented classical heritage and maintaining a foundational level of mathematical literacy

Translation from Arabic to Latin

The intellectual gap:

- After the fall of Rome, much of the advanced Greek and Indian mathematical knowledge was lost to Western Europe
- Meanwhile, the Islamic world had preserved, translated, and significantly advanced this knowledge (as we've seen!)

Translation centres (11^{th}–13^{th} centuries CE):

- **Toledo, Spain**: After its reconquest from Muslim rule, became a major hub for translation
- **Sicily**: Another important point of cultural and intellectual exchange
- Scholars (often Christian, some Jewish) travelled to these centres to learn Arabic and translate texts

Translation from Arabic to Latin

Key translators:

- **Adelard of Bath** (~1080–1152 CE):
  - Translated Euclid's "Elements" (first Latin translation from Arabic)
  - Introduced Hindu-Arabic numerals to England
- **Gerard of Cremona** (~1114–1187 CE):
  - The most prolific translator, translating over 70 Arabic books into Latin
  - Included Ptolemy's "Almagest", Al-Khwarizmi's "Algebra", and works by Archimedes, Hippocrates, and others

Re-awakening European thought: This massive influx of sophisticated mathematical, astronomical, and philosophical texts from the Islamic world was arguably the single most important factor in the subsequent intellectual awakening of Europe

The impact of Hindu-Arabic numerals

A slow revolution:

- The Hindu-Arabic numeral system (including zero and place value) was known in Europe from the 10^{th}–11^{th} centuries (Gerbert of Aurillac, Adelard of Bath)
- However, its adoption was **very slow** due to:
  - Entrenched use of Roman numerals and abacus
  - Distrust of foreign methods ("devil's arithmetic")
  - Lack of a centralised authority to mandate change

The impact of Hindu-Arabic numerals

Leonardo Fibonacci (~1175–1250 CE):

- Born in Pisa, Italy (a major trading centre)
- Travelled extensively in the Mediterranean (North Africa, Middle East), learning mathematical methods
- His most famous work: ***Liber Abaci***, Book of calculation (1202)

The impact of Hindu-Arabic numerals

***Liber Abaci*'s impact**:

- A comprehensive treatise on arithmetic and algebra extensively using Hindu-Arabic numerals
- Introduced the system to a wider European audience through numerous practical examples (commerce, weights, measures, proportions)
- Demonstrated the system's superior efficiency compared to Roman numerals for merchants and bankers
- Also contained the famous "Fibonacci sequence" (though not its primary purpose)

The impact of Hindu-Arabic numerals

A page from Fibonacci's Liber Abaci

Laying the foundations for modern arithmetic: Fibonacci's "Liber Abaci" was a critical step in the eventual triumph of the Hindu-Arabic numeral system in Europe, revolutionising calculation

The impact of Hindu-Arabic numerals

The first part of "Liber Abaci" introduces the Hindu–Arabic system, its arithmetic, and methods for converting between different representations

The second section presents examples from commerce, such as conversions of currency and measurements, and calculations of profit and interest

The third section discusses a number of mathematical problems; for instance, it includes the Chinese remainder theorem, perfect numbers and Mersenne primes as well as formulas for arithmetic series. Another example in this chapter involves the growth of a population of rabbits, where the solution requires generating a numerical sequence, known as Fibonacci since then

The fourth section derives approximations, both numerical and geometrical, of irrational numbers such as square roots

The Renaissance

What was the Renaissance? A new spirit of inquiry (~14th–17th century)

- Literally "rebirth" (French)
- Period of intense cultural, artistic, political, and scientific "rebirth" in Europe, following the Middle Ages
- Began in **Italy** (14th century, Florence), then spread across Europe

A new intellectual climate:

- **Humanism:** Renewed focus on human potential, achievements, and the study of classical (Greek and Roman) texts directly
- **The printing press** (Gutenberg, ~1440): Revolutionised the dissemination of knowledge, making mathematical texts more widely available with respect to previous times
- **Emphasis on observation & practicality**: Growing interest in describing and understanding the natural world, engineering, commerce, and art with precision

Mathematics in art

The challenge for artists:

- How to represent three-dimensional space and objects accurately on a two-dimensional surface?
- Medieval art often lacked realistic depth

Filippo Brunelleschi (~1377–1446 CE):

- Florentine architect and engineer
- Credited with demonstrating the principles of **linear perspective** around 1415
- Used mathematical principles (geometry, optics) to create a consistent illusion of depth

Piero della Francesca (~1415–1492 CE):

- Painter and mathematician
- Applied rigorous mathematical principles to his paintings, showcasing perfect perspective
- Wrote treatises on perspective and solid geometry

Mathematics in art

Leon Battista Alberti (~1404–1472 CE):

- Architect and theorist
- Codified the rules of linear perspective in his treatise *De pictura* (On Painting, 1435)
- Described how to use a single "vanishing point" to create proportional recession

Albrecht Dürer (~1471–1528 CE):

- German painter, print maker, and theorist
- Wrote influential books on geometry and perspective, making these ideas accessible
- Developed methods for drawing various objects in perspective

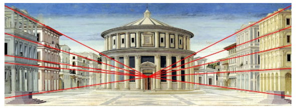

Diagram illustrating principles of linear perspective

A practical Renaissance for geometry: The quest for realistic representation in art spurred a renewed interest in and practical application of geometry, laying groundwork for later advancements in projective geometry

Renaissance geometry

Beyond perspective:
Other geometric explorations

- While linear perspective was a major driver, Renaissance mathematicians also continued to study classical Euclidean geometry
- **Albrecht Dürer** (~1471–1528 CE):
  - Beyond art, his books (*Underweysung der Messung*, 1525) systematically explored practical geometry
  - Covered polygons, curves, solids, and methods for projections (**descriptive geometry**)

Albrecht Dürer's geometric constructions

- Emphasis shifted from purely abstract proofs (though still valued) to precise construction and application

The Italian algebra school

The mathematical challenge:

- For centuries, mathematicians had systematically solved linear and quadratic equations (e.g., from Babylonians, Greeks, Islamic scholars)
- The **cubic equation** ($ax^3 + bx^2 + cx + d = 0$) remained intractable
- No general algebraic formula was known for its roots.

The "Scuola d'Algebra" (School of Algebra):

- Flourished in 16^{th} century Italy, driven by public challenges and intellectual rivalry
- These mathematicians were often independent scholars, teachers, and sometimes associated with universities
- The quest to solve the cubic became a highly competitive secret

The Italian algebra school

Key cities of the Italian Algebra School

Why the secrecy?

- Knowing how to solve cubics offered prestige and financial advantage (e.g., solving problems for patrons, winning public debates)
- Formulas were kept secret to maintain competitive edge

A period of intense discovery: The 16^{th} century in Italy saw a dramatic shift in algebra, moving beyond classical geometric methods to seek general algebraic solutions

The cubic equation

Scipione del Ferro (~1465–1526 CE):
The first solver

- Professor at the University of Bologna
- Secretly discovered a general method to solve one form of the cubic equation: $x^3 + px = q$
- Did not publish his method, but shared it with a few students, including Antonio Fiore

Niccolò Fontana Tartaglia (~1500–1557 CE):
The public solver

- Re-discovered the solution to the cubic $x^3 + px = q$ (and $x^3 + q = px$)
- Famous for winning public mathematical contests, notably against Fiore, by solving cubics
- Initially refused to share his secret formula

Niccolò Fontana Tartaglia

The cubic equation

Gerolamo Cardano (~1501–1576 CE):
The publisher

- Physician, astrologer, and brilliant mathematician
- Persuaded Tartaglia to share his secret formula under an oath of secrecy
- Later discovered (through del Ferro's papers) that Tartaglia was not the first, feeling released from his oath
- Published the formula in his groundbreaking book, **Ars Magna** (The Great Art), in 1545

Gerolamo Cardano

The cubic equation

Cardano's formula (for $x^3 + px = q$):

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

This formula revealed a profound, complex challenge for numbers!

The quartic equation

Ludovico Ferrari (~1522–1565 CE): The quartic solver

- Cardano's student
- Building on the cubic solution, he discovered a method to solve the **quartic equation** ($ax^4 + bx^3 + cx^2 + dx + e = 0$)
- His method involved reducing the quartic to a cubic, thus relying on Cardano's formula. The reduction is accomplished by a change of variable $x = u - \frac{b}{4a}$ which yields $u^4 + Au^2 + Bu + C = 0$, a *depressed quartic* equation. If $B = 0$ the equation is *biquadratic* and thus easily solved. Otherwise a number of quite involved substitutions is performed to obtain a cubic equation whose solutions allow the generate the roots of the depressed quartic
- This was also published in Cardano's *Ars Magna*

The quartic equation

The "irreducible case" of the cubic:

- A major problem arose when applying Cardano's cubic formula to equations like $x^3 - 15x = 4$
- The formula yields: $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$
- This involves square roots of negative numbers, even though the equation has three **real** roots ($x = 4, x = -2 + \sqrt{3}, x = -2 - \sqrt{3}$)
- This forced mathematicians to confront these "imaginary" quantities

The quartic equation

Rafael Bombelli (~1526–1572 CE):
Embracing the impossible

- Engineer and mathematician
- In his *L'Algebra* (1572), he systematically introduced and operated with square roots of negative numbers
- He showed how these "impossible" numbers could combine to yield real solutions for the cubic equation
- This was the true **birth of complex numbers** as a necessity for algebra

From algebraic necessity to new numbers: The solution of the cubic equation, paradoxically, forced mathematicians to acknowledge and work with numbers previously deemed impossible, fundamentally expanding the concept of "number"

The quartic equation

For example, Bombelli's technique in calculating

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

assumes a and b to be unknowns and since

$$(a + bi)^3 = a(a^2 - 3b^2) + b(3a^2 - b^2)i$$
$$(a - bi)^3 = a(a^2 - 3b^2) - b(3a^2 - b^2)i$$

thus, assuming $\sqrt[3]{2 + \sqrt{-121}} = a + bi$ and $\sqrt[3]{2 - \sqrt{-121}} = a - bi$, a root is $x = a + bi + a - bi = 2a$.
A simple check yields $a = 2$ and $b = 1$ as a possible solution for the pair of equations $a(a^2 - 3b^2) = 2$ and $b(3a^2 - b^2) = 11$, that is, $x = 4$

The dawn of symbolic algebra

François Viète

François Viète (~1540–1603 CE):

- Lawyer and advisor to French kings, but a dedicated mathematician
- Recognised the limitations of rhetorical (word-based) and syncopated (abbreviated) algebra
- His major work: *In Artem Analyticam Isagoge* (Introduction to the Analytical Art, 1591)

The dawn of symbolic algebra

The revolution of symbolic algebra:

- Introduced the use of **letters for both known quantities** (coefficients) (vowels: A, E, I, O, U) **and unknown quantities** (variables) (consonants: B, C, D...)
- Allowed for the expression of **general formulas** and theorems, not just solutions to specific numerical problems
- Example: Instead of "the cube plus six times the side equals twenty", write $B^3 + 6B = 20$ (if B is the unknown)
- This was a massive conceptual leap, abstracting algebra from specific numbers and from a strict geometrical interpretation

The dawn of symbolic algebra

Viète's contributions beyond notation:

- He developed a number of relationships between the roots and coefficients of polynomial equations, summarised in the modern *Viète formula*

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}$$

  for every $k = 1, 2, \ldots, n$
- Used his new notation to simplify and systematise many algebraic and trigonometric problems
- He is also known for his work on computing π (Viète's formula)

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2+\sqrt{2}}}{2} \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

A foundation for modern algebra: Viète's symbolic notation was pivotal. It transformed algebra into a powerful abstract tool, enabling the work of Descartes, Fermat, and the later development of calculus

Early scientific connections

Mathematics: The language of the universe

- A profound philosophical shift occurred: the belief that the universe was ordered mathematically
- This idea, present in ancient Greece, was powerfully re-asserted
- Mathematicians were central to the new science

Early scientific connections

Early scientific revolution figures:

- **Nicolaus Copernicus** (~1473–1543 CE): Used advanced geometry and trigonometry to develop his heliocentric model
- **Johannes Kepler** (~1571–1630 CE): Derived laws of planetary motion from astronomical observations using geometric and algebraic methods
- **Galileo Galilei** (~1564–1642 CE): Applied mathematical principles to physics, describing motion with the aid of algebraic formulas and geometric diagrams

From abstract to applied: Mathematics was increasingly seen as the essential tool for understanding and describing the physical world, driving a convergence of theory and experiment

The invention of logarithms

The problem: Astronomical calculations

- By the late Renaissance, astronomical data and calculations (multiplication of large numbers, division, finding roots) were becoming incredibly complex and time-consuming
- The need for a way to simplify these computations was immense

The invention of logarithms

John Napier, inventor of logarithms

John Napier (~1550–1617 CE):
The Scottish Baron

- Published his work *Mirifici Logarithmorum Canonis Descriptio* (Description of the Wonderful Canon of Logarithms) in 1614
- Developed logarithms based on a mechanical concept of points moving along lines
- His initial logarithms were "Napierian" (base $1/e$)

The invention of logarithms

Jost Bürgi (~1552–1632 CE):
The Swiss clock maker

- Independently developed logarithms around the same time as Napier
- His tables were published later (1620)

Henry Briggs (~1561–1630 CE):
Towards base 10

- Collaborated with Napier to convert his logarithms to the more practical **base 10**
- This form became the standard for practical computation

The ultimate calculator of its time: Logarithms transformed calculation, making complex multiplication and division into simple addition and subtraction. They were indispensable for astronomy, navigation, and engineering for centuries

Conclusion: From recovery to revolution

Laying the groundwork for modern mathematics

- **The Middle Ages**:
  - A period of crucial **preservation** of classical texts
  - The vital **translation movement** from the Islamic world re-introduced advanced knowledge to Europe
  - Gradual adoption of **Hindu-Arabic numerals** (Fibonacci)
- **The Renaissance**:
  - Fostered by a new spirit of humanism, observation, and practicality (printing press)
  - **Linear perspective** in art spurred geometric advancements
  - Dramatic breakthroughs in **algebra**: solving cubic and quartic equations (Italian School)
  - The necessary emergence of **complex numbers** (Bombelli)
  - The invention of **symbolic algebra** (Viète), a fundamental conceptual leap
  - The revolutionary computational tool of **logarithms** (Napier)

Conclusion: From recovery to revolution

Laying the groundwork for modern mathematics

- **Mathematics as the Language of Science**:
  - Became central to understanding the natural world, fuelling the nascent Scientific Revolution (Copernicus, Kepler, Galileo)

This period transformed mathematics from a fragmented heritage into a vibrant, abstract, and powerfully applied discipline, ready for the explosion of analytical geometry and calculus in the 17th century

References

The main sources for this lecture are

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- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Dirk J. Struik* A Concise History of Mathematics, Fourth revised edition, Dover Publications (1987)

Our account omitted many important mathematicians, and skipped over many important themes. We recommend *Luca Pacioli*, Summa de arithmetica geometria proportioni et proportionalità (1494), which is a sort of encyclopedia of Mathematics of its time. It was a very expensive book (Leonardo da Vinci paid 16 *soldi* for his copy), but superbly complete

History of Mathematics: Lecture 8

Syllabus:

Mathematics in the 17th century

- The dawn of modernity

Revolutionary ideas and foundational tools

A pivotal era:

- The 17th century (roughly 1600–1700 CE) stands as a watershed moment in the history of mathematics
- It saw the invention of radically new mathematical fields and powerful tools that remain central to science and engineering today
- It was an era marked by profound intellectual shifts and unprecedented scientific discovery

The scientific revolution

Astronomical transformation:

- Building on Copernicus: **Johannes Kepler's** laws of planetary motion (early 17th century)
  - Described elliptical orbits and variable speeds
  - Required more sophisticated mathematical tools

Key figures of the scientific revolution: Kepler, Newton, Galileo

- **Galileo Galilei's** telescopic observations and work on mechanics
  - Challenged Aristotelian physics
  - Emphasised observation and mathematical description of natural phenomena

The scientific revolution

The birth of modern physics:

- The century culminates with **Isaac Newton's** monumental work, *Principia Mathematica* (1687)
  - Unified celestial and terrestrial mechanics under universal laws
  - Expressed these laws in the language of mathematics
- This required tools to describe **change, motion, rates, and accumulation** — setting the stage for calculus

Philosophical currents:

- **Rationalism** (e.g., René Descartes): Emphasised reason, deduction, and mathematical clarity as paths to knowledge
- **Empiricism** (e.g., Francis Bacon, John Locke): Stressed observation and experimentation, often leading to data that needed mathematical analysis
- These philosophies promoted mathematics as the **fundamental language of nature**

The rise of scientific societies

The Royal Society of London

Accademia del Cimento in Florence

Institutionalisation of Science:

- Establishment of formal scientific societies across Europe: for example
  - **Accademia del Cimento** (Florence, 1657)
  - **Royal Society of London** (1660)
  - **Académie des Sciences** (Paris, 1666)
- Provided platforms for collaboration, publication, peer review, and funding
- Fostered a competitive yet collaborative environment for mathematical and scientific discovery

The need for computation

The growing demand for calculation:

- The new astronomy and physics required increasingly complex and precise calculations (e.g., planetary orbits, trajectories)
- Navigation and commerce also demanded better tools for calculation
- While logarithms were a great leap, there was a drive for even more efficient methods and theoretical foundations for understanding continuous change
- This practical need, plus philosophical inquiry, set the stage for calculus

A century primed for mathematical revolution: The intellectual climate, scientific challenges, and institutional support of the 17th century created an unprecedented environment for mathematical innovation

René Descartes

A giant of thought (~1596–1650 CE):

- French philosopher, scientist, and mathematician
- Often called the “Father of Modern Philosophy”.
- His famous dictum: “*Cogito, ergo sum*” (“I think, therefore I am”)
- Emphasised **rationalism** and the pursuit of certainty through clear and distinct ideas, often inspired by mathematics

René Descartes

René Descartes

The *Discourse on Method* (1637):

- His seminal philosophical work
- Included three appendices, one of which was ***La Géométrie*** (The Geometry)
- This appendix, just 100 pages, profoundly transformed mathematics

Motivation for *La Géométrie*:

- To demonstrate the power of his new philosophical method by applying it to mathematics
- To solve classical geometric problems using purely algebraic techniques
- To make geometry more systematic and less reliant on individual ingenuity

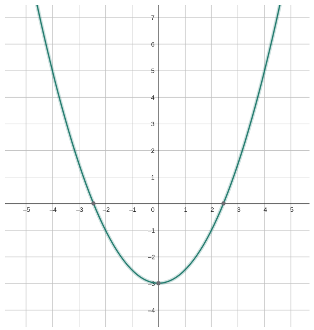
René Descartes

Descartes’ vision:

- To unify all scientific knowledge under a single, universal method based on mathematical reasoning
- Algebra, with its precise rules, offered the perfect tool for analysing and solving geometric problems

The power of methodical thought: Descartes sought to bring the clarity and certainty of arithmetic and algebra into geometry, creating a new way to understand shapes and spaces

The Cartesian coordinate system



Illustrating the Cartesian coordinate system

The revolutionary idea:

- Descartes’ key insight: Any point in a plane can be precisely located by a pair of numbers (coordinates)
- Any curve or geometric shape can then be represented by an **algebraic equation**
- Conversely, any algebraic equation in two variables can be represented as a geometric curve

Unifying algebra and geometry

Setting up the system (Descartes' approach):

- Started with a single reference line (the x-axis).
- Distinguished lengths (variables x, y, z) from fixed magnitudes (constants a, b, c)
- Used equations to define curves. For example, a circle centred at the origin: $x^2 + y^2 = r^2$
- While not identical to modern orthogonal coordinates (he often used oblique axes), the fundamental principle was there

Example: Solving a geometric problem algebraically

- Instead of complex geometric constructions, problems could be translated into equations
- Find the intersection of two lines: Solve their two algebraic equations simultaneously
- Determine properties of a curve: Analyse its defining equation

Unifying algebra and geometry

Impact of this unification:

- Allowed for the application of powerful algebraic techniques to solve geometric problems that were previously very difficult or impossible
- Provided a systematic, analytical method for geometry, reducing reliance on visual intuition for complex cases
- Laid the essential groundwork for the development of **calculus**, which inherently deals with curves and their properties (tangents, areas)

A synthesis that defined modern mathematics: Analytical geometry bridged the gap between two previously distinct branches of mathematics, creating a powerful new analytical tool

Pierre de Fermat

A contemporary genius (~1601–1665 CE):

- French lawyer and amateur mathematician
- Often called the "Prince of Amateurs" due to his profound contributions despite not being a professional academic
- Worked in relative isolation, communicating primarily through letters

Pierre de Fermat

Independent discovery of analytical geometry:

- Developed similar ideas to Descartes regarding the relationship between equations and curves
- His manuscript, *Ad locos planos et solidos isagoge* (Introduction to Plane and Solid Loci), was written around 1636, roughly concurrent with Descartes' work
- It was published posthumously in 1679

Pierre de Fermat

Fermat's other contributions (prelude to calculus):

- Developed a method for finding the **maxima and minima** of functions by setting the derivative to zero (a precursor to differential calculus)
- Found methods for calculating **areas under curves** (a precursor to integral calculus)
- His work on tangents and quadratures directly influenced Newton

Pierre de Fermat

Fermat's legacy beyond analytical geometry:

- **Number theory**: Famously known for "Fermat's Last Theorem" and his work on prime numbers and divisibility
- **Probability theory**: Co-founded probability theory with Pascal through their correspondence on gambling problems
- **Optics:** Fermat's principle of least time

A quiet giant whose ideas resonated: Fermat's isolated brilliance led to discoveries that paralleled and often anticipated those of his contemporaries, profoundly influencing the trajectory of 17^{th} century mathematics

The problems leading to calculus

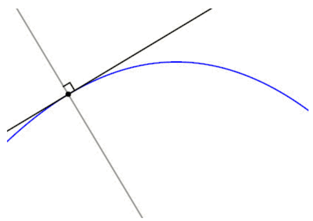
The mathematical challenges of change and accumulation:

- The Scientific Revolution demanded tools to describe:
  - **Rates of change** (e.g., velocity, acceleration)
  - **Slopes of curves** at any point (e.g., trajectory of a projectile)
  - **Accumulation** of quantities (e.g., total distance travelled)
  - **Areas and volumes** of complex shapes
- Traditional Euclidean geometry and algebra were insufficient for these dynamic problems

The problems leading to calculus

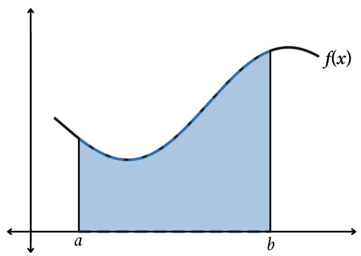
The **Problem of tangents** (Differential Calculus):

- How to find the slope of a curve at any given point?
- For a circle, it was known, but for a parabola, ellipse, or any arbitrary curve?
- This problem relates directly to instantaneous rate of change



The challenge of finding the tangent to a curve

The problems leading to calculus



The challenge of finding the area under a curve

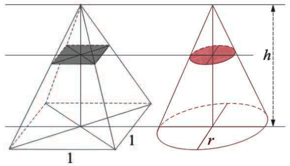
The **Problem of areas** or **Quadratures** (Integral Calculus):

- How to find the area bounded by a curve (or curves)?
- For simple shapes (rectangles, triangles), formulas existed, but for complex or irregular curves?
- This problem relates to accumulation and total quantity
- Ancient Greek methods (like Archimedes' method of exhaustion) were limited and cumbersome

Early precursors to calculus

Bonaventura Cavalieri (~1598–1647 CE):
Method of indivisibles

- Italian mathematician, heavily influenced by Archimedes
- His *Geometria Indivisibilibus Continuorum Nova Quadam Ratione Promota* (1635) introduced the **method of indivisibles**



Cavalieri's principle for volumes

- Conceived of areas as made up of an infinite number of parallel line segments and volumes as made up of an infinite number of parallel planes
- Enabled calculation of areas and volumes for complex shapes (e.g., area under a parabola, volume of a sphere)
- A powerful heuristic, though lacked the rigour of later calculus

Early precursors to calculus

Pierre de Fermat (revisited): Maxima, minima, and tangents

- As mentioned before, Fermat developed a method for finding maxima and minima of polynomial functions
- His technique involved considering a small change in the variable, similar to the idea of a derivative
- He also used this method to find tangents to curves
- His work was very close to the concept of differentiation

Early precursors to calculus

Isaac Barrow (~1630–1677 CE): Newton's teacher

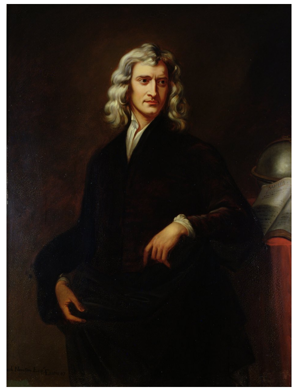
- English mathematician and theologian, first Lucasian Professor of Mathematics at Cambridge (position later held by Newton)
- Made significant progress on the problems of tangents and areas
- Crucially, he observed that finding the area under a curve and finding the tangent to a curve were **inverse operations**
- This insight was a precursor to the Fundamental Theorem of Calculus

From heuristic methods to systematic theory: These early contributions demonstrated powerful techniques, paving the way for the generalised, systematic framework of calculus

Isaac Newton

An unparalleled genius
(~1643–1727 CE):

- English physicist, mathematician, astronomer, alchemist, theologian
- Began developing his version of calculus, which he called **fluxions**, during the plague years (1665–1666), in isolation
- Very reluctant to publish his mathematical methods; insights appeared in his physics works



Isaac Newton

Isaac Newton

Newton's conceptual framework:

- Viewed quantities as generated by **continuous motion** (e.g., a line generated by a moving point, an area by a moving line)
- **Fluents**: The flowing quantities (e.g., x, y)
- **Fluxions**: The instantaneous rates of change of these fluents (e.g., $\dot{x}$, $\dot{y}$)
- Used infinite series extensively

The Fundamental Theorem of Calculus:

- Newton rigorously proved the inverse relationship between differentiation (finding fluxions) and integration (finding fluents from fluxions)
- This formalised Barrow's earlier insight

Isaac Newton

Application to Physics: *Principia Mathematica* (**1687**):

- His magnum opus, describing universal gravitation and laws of motion
- The entire work is built upon his new mathematical methods (fluxions and geometrical arguments)
- Used calculus to:
  - Explain planetary orbits
  - Model the motion of objects under forces
  - Predict phenomena like tides

Calculus as the engine of the universe: Newton's calculus provided the mathematical language to describe the dynamic, changing universe, ushering in modern physics

Gottfried Wilhelm Leibniz

Gottfried Wilhelm Leibniz

A universal thinker (~1646–1716 CE):

- German philosopher, mathematician, logician, and diplomat
- Independently developed his version of calculus from 1673 onwards, inspired by reading Pascal's work
- Published his first paper on differential calculus in 1684 (*Nova Methodus pro Maximis et Minimis...*)

Gottfried Wilhelm Leibniz

Leibniz's conceptual framework and notation:

- Focused on infinitesimals (infinitely small quantities) and differences
- His notation is still used today and is often considered more intuitive
  - $\frac{dy}{dx}$: For differentiation (rate of change of y with respect to x)
  - $\int y\,dx$: For integration (summing up infinitesimal areas)
- Emphasised the "algorithm" and rules of manipulation, making calculus more accessible

Gottfried Wilhelm Leibniz

The calculus priority dispute:

- A bitter and prolonged controversy erupted between supporters of Newton and Leibniz over who invented calculus first
- Modern historical consensus: Both invented calculus independently
- Newton's discovery was earlier (1660s), but Leibniz's publication was earlier (1684) and his notation superior

The dual birth of a transformative tool: The independent development by Newton and Leibniz underscores the readiness of the mathematical world for calculus, which provided the essential tool for the new physics and beyond

The birth of probability theory

From gambling to mathematics:

- The formal study of probability arose not from scientific inquiry, but from **gambling problems**
- Antoine Gombaud, Chevalier de Méré, a French writer and avid gambler, posed questions about fair division of stakes in interrupted games

The problem of points:

- If two players agree to play a series of games, and the game is interrupted before completion, how should the stakes be divided fairly, based on the current score?
- This problem had puzzled mathematicians for centuries

Blaise Pascal

Blaise Pascal (~1623–1662 CE):

- French mathematician, physicist, inventor, writer, and theologian
- Independently developed his ideas on probability
- Also known for Pascal's Triangle (again!), which relates to binomial coefficients and combinations, foundational for probability

Blaise Pascal

The birth of probability theory

Correspondence with Pierre de Fermat (1654):

- Pascal and Fermat exchanged a series of letters discussing the Problem of Points and its solution
- This correspondence laid the **foundations of modern probability theory**
- They introduced concepts like expected value and combinatorial analysis to solve the problem systematically

The birth of probability theory

Christian Huygens (~1629–1695 CE):

- Dutch mathematician, physicist, astronomer
- Wrote the first formal treatise on probability: *De ratiociniis in ludo aleae* (On Reasoning in Games of Chance, 1657), essentially based on the Pascal-Fermat correspondence

From chance to science: Probability theory, born from practical gambling questions, would grow into an indispensable tool for statistics, risk assessment, science, and social sciences

Other 17th century advances

Number theory: Fermat's unproven theorems

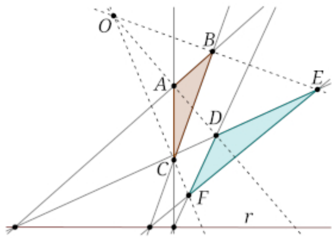
- **Pierre de Fermat** (revisited again!): A key figure in number theory, posing many theorems without publishing proofs
- **Fermat's Last Theorem**: "No three positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2".
  - Famously stated in a margin with a note that he had a "truly marvellous proof. . . which this margin is too narrow to contain!.
  - Remained unproven for over 350 years, until Andrew Wiles in 1994
- Also worked on prime numbers, divisibility, and integer solutions to equations (Diophantine equations)

Other 17th century advances

Projective geometry:

- **Gérard Desargues** (~1591–1661 CE): French architect and engineer
- Laid the foundations of projective geometry, focusing on properties of figures that are invariant under projection (relevant to perspective)

The theorem bearing his name recalls Pappus's result:
Two triangles are in perspective *axially* if and only if they are in perspective *centrally*

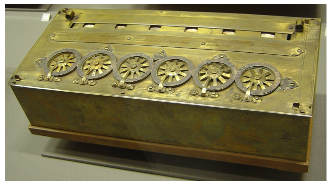


Other 17th century advances

Refining logarithms & computation:

- After Napier and Briggs, logarithm tables were refined and widely adopted
- **William Oughtred** (~1574–1660 CE): Invented the **slide rule** (1622), a mechanical analog computer based on logarithms

- **Blaise Pascal** (revisited): Invented the **Pascaline** (1642), an early mechanical calculator
- These tools drastically reduced the labour of complex arithmetic for scientists and engineers



Pascal's mechanical calculator, the Pascaline

A century of tools and concepts: The 17th century provided not only powerful new branches of mathematics but also practical computational aids, setting the stage for industrial and scientific revolutions

Conclusion: A transformative century

From ancient problems to modern analytical power

- **Intellectual context**: Fuelled by the Scientific Revolution and new philosophical paradigms (Rationalism, Empiricism)
- **The unification of mathematics**:
  - **Analytical geometry** (Descartes, Fermat) merged algebra and geometry, creating a systematic framework for curves and equations
- **The mathematics of change**:
  - The independent invention of **calculus** (Newton, Leibniz) provided the indispensable tools for understanding rates of change, motion, accumulation, and more
  - Solved the "problem of tangents" and "problem of areas"

Conclusion: A transformative century

From ancient problems to modern analytical power

- **New fields emerge**:
  - The formal birth of **probability theory** (Pascal, Fermat, Huygens).
  - Advances in number theory and projective geometry
- **Practical impact**: Development of computational aids (slide rule, Pascaline) and the application of mathematics as the core language of physics

The 17th century fundamentally reshaped mathematics, transitioning it into its modern analytical form, equipped with the tools and concepts to tackle the most complex scientific and engineering challenges for centuries to come

References

The main sources for this lecture are

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Dirk J. Struik* A Concise History of Mathematics, Fourth revised edition, Dover Publications (1987)

Also, the classical although not always accurate *Eric T. Bell*, Men of Mathematics, Touchstone (1986) is worth reading

For fun, we highlight *Michael White*, Isaac Newton: The Last Sorcerer, Fourth Estate (2012), which explores how magic influenced the birth of Science

History of Mathematics: Lecture 9

Syllabus:

Mathematics in the 18th century

- The age of Euler

The reign of analysis

The 18th century: An era of rapid mathematical growth

- Follows the foundational discoveries of the 17th century (analytical geometry, calculus, probability)
- Characterised by the **rapid development and systematisation of calculus and its applications**
- Often referred to as the **Age of Analysis**

Key focus for this lesson:

- The legacy of the **Bernoulli family**
- The unparalleled contributions of **Euler**, who dominated the century
- The consolidation and widespread application of calculus

The Enlightenment and its impact

The age of reason:

- A powerful intellectual and cultural movement across Europe
- Emphasised **reason, individualism, and scepticism** over tradition and faith
- Belief in the power of human intellect to understand and improve the world

Enlightenment thinkers valuing reason and science

The Enlightenment and its impact

Mathematics as the ultimate rational endeavour:

- The success of Newton's mathematically formulated laws of physics profoundly influenced Enlightenment thinkers
- Mathematics was seen as the purest form of rational thought, capable of uncovering universal truths
- Led to a strong push for **systematisation, clarity, and logical deduction** in all fields, including mathematics

The Enlightenment and its impact

The role of academies and patronage:

- **Royal Academies** (Paris, Berlin, St. Petersburg) became dominant centres of research and publication
- Provided salaried positions for leading mathematicians, as it was the case for Euler and Lagrange
- Organised prize competitions for solving challenging mathematical problems
- Fostered international communication and collaboration (despite rivalries)

A fertile ground for mathematical growth: The Enlightenment's belief in reason, coupled with institutional support, created an environment ripe for rapid mathematical advancement and application

The Bernoulli family

Expanding the boundaries of calculus: **A family affair**:

- A remarkable family from Basel, Switzerland, that produced numerous outstanding mathematicians over several generations
- Most prominent: **Jacob Bernoulli** (1654–1705), **Johann Bernoulli** (1667–1748), and Johann's son **Daniel Bernoulli** (1700–1782)
- Key figures in the early 18^{th} century, extending and applying the newly invented calculus
- Known for both collaboration and intense rivalry!

Their role in the age of analysis:

- Active in the European academies, contributing to prize problems
- Applied calculus to a wide range of physical problems
- Pushed the boundaries of differential equations, probability, and geometry

Jacob Bernoulli

Probability theory:

- Published *Ars Conjectandi* (The Art of Conjecturing) posthumously in 1713
- Contained the first substantial treatise on probability theory since Huygens
- Introduced the **Law of Large Numbers**, formally proving that as the number of trials increases, the observed frequency of an event approaches its theoretical probability: given a collection of independent and identically distributed samples from a random variable X_i with finite mean μ, the sample average converges almost surely to the expected value

Jacob Bernoulli

$$\mathrm{P}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i=\mu\right)=1$$

Jacob Bernoulli

Infinite series:

- Pioneered the study of convergence of infinite series
- Introduced **Bernoulli numbers** (B_n), which appear in the Taylor series expansion of trigonometric functions and in formulas for sums of powers of integers (recall Seki Takakazu)

$$\tan(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} \qquad |x|<\frac{\pi}{2}$$

Jacob Bernoulli

The logarithmic spiral (Spira Mirabilis):

- Famously studied the properties of the **logarithmic spiral** (also known as the growth spiral or spira mirabilis) $r=ae^{k\theta}$ in polar coordinates with a,k constants
- Discovered that its shape is invariant under changes in scale (dilations)
- Was so fascinated by its properties that he requested it be engraved on his tombstone with the inscription *Eadem mutata resurgo* ("Though changed, I rise again the same")
- Related to the Fibonacci sequence and the golden ratio.

A pioneer in many fields: Jacob Bernoulli's work laid critical groundwork for modern probability and the study of infinite processes

Johann Bernoulli

Johann Bernoulli

A prolific and influential teacher:

- Younger brother of Jacob, and perhaps even more influential through his teaching and extensive correspondence (e.g., with Leibniz)
- Mentored many great mathematicians, including his son Daniel and **Leonhard Euler**

Johann Bernoulli

The brachistochrone problem (1696):

- Posed as a challenge problem:
  "What is the curve along which a particle, acted on only by gravity, will fall in the shortest time between two points"?
- This problem was crucial for the development of the **calculus of variations** (finding functions that optimise integrals)
- The solution is a **cycloid** (the curve traced by a point on the rim of a rolling wheel)

Johann Bernoulli

L'Hôpital's rule (and the controversy):

- Johann taught calculus to Guillaume de l'Hôpital, under an agreement that he could publish Bernoulli's discoveries
- L'Hôpital published the first calculus textbook: *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (1696)
- This book included "L'Hôpital's rule" for evaluating limits of indeterminate forms (like $0/0$ or ∞/∞)

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

- It was later revealed that the rule (and much of the book) was a direct result of Johann Bernoulli's work and lessons

A master of analytical problem-solving: Johann Bernoulli pushed the boundaries of calculus and inspired the next generation of analysts, despite his contentious nature

Daniel Bernoulli

Daniel Bernoulli

Son of Johann, and a universal scientist:

- One of the most prominent scientists of his time, winning 10 Grand Prizes from the Paris Academy of Sciences (often in competition with his father and brother!)
- Known for applying mathematical principles to a wide range of physical problems

Daniel Bernoulli

Hydrodynamics (1738):

- His magnum opus, laying the foundation for modern fluid mechanics
- Introduced **Bernoulli's principle**, which describes the relationship between fluid velocity and pressure (e.g., how an aeroplane wing generates lift)
- Applied calculus to describe the flow of water and other fluids

Probability in physics and statistics:

- Applied probability to questions of risk and expectation (e.g., the St. Petersburg paradox)
- Made early contributions to mathematical statistics, particularly in modelling errors and combining observations

Daniel Bernoulli

St. Petersburg paradox:
A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake begins at 2 euros and is doubled every time tails appears. The first time heads appears, the game ends and the player wins whatever is the current stake, that is, the player wins 2^{k+1} euros, where k is the number of consecutive tails tosses
What would be a fair price to pay the casino for entering the game?

The paradox arises because the expectation is ∞

Daniel Bernoulli

Other areas of influence:

- **Vibrating strings**: Pioneering work on the mathematics of vibrating strings, contributing to the development of Fourier series
- **Kinetic theory of gases**: Early ideas on the microscopic nature of gases and their pressure
- **Medicine**: Applied statistical methods to smallpox inoculation, an early example of medical statistics

Bridging mathematics and the physical world: Daniel Bernoulli exemplified the Enlightenment ideal of applying rigorous mathematical analysis to understand and explain natural phenomena

Leonhard Euler

Life and career (1707–1783):

- Born in Basel, Switzerland, just like the Bernoullis (and a student of Johann Bernoulli)
- Spent most of his career in **St. Petersburg, Russia**, and **Berlin, Prussia**, at the respective Academies of Sciences
- Lost sight in one eye early in his career, and became almost completely blind in his later years, yet his productivity barely slowed
- Published over 800 books and papers, encompassing almost every field of mathematics and much of physics. His collected works fill over 80 volumes!

Leonhard Euler

Leonhard Euler

A unifying force in 18th century mathematics:

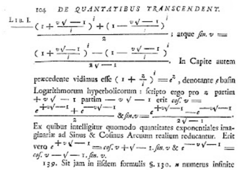
- Took the ideas of Newton and Leibniz and developed them into the modern form of calculus and analysis
- Systematised vast areas of mathematics that were previously disparate or underdeveloped
- Authored highly influential textbooks that shaped mathematical education for centuries

His enduring legacy:

- His work profoundly influenced mathematics, physics, and engineering
- Much of the notation and many of the methods we use today stem directly from Euler
- Considered by many to be one of the greatest mathematicians of all time

The architect of modern mathematics: Euler's unique blend of insight, analytical power, and incredible productivity laid the bedrock for mathematical analysis as we know it

Euler's standardised notation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Euler's clarity in mathematical expression

Clarity and consistency:

- Before Euler, mathematical notation was often inconsistent and varied between mathematicians
- Euler's clear, logical, and expressive notation rapidly gained widespread adoption, significantly aiding communication and development

Euler's standardised notation

Some of his enduring notational contributions:

- π: For the ratio of a circle's circumference to its diameter (adopted 1736)
- e (Euler's number): For the base of the natural logarithm (adopted 1727, widely used by 1748)
- i: For the imaginary unit ($\sqrt{-1}$) (adopted 1777)
- $f(x)$: For function notation (adopted 1734)
- $\sum$: For summation (adopted 1755)
- Δ: For finite differences
- Standard notation for trigonometric functions ($\sin, \cos, \tan$)

Euler's standardised notation

Why notation matters:

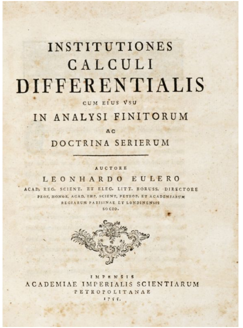
- Good notation simplifies ideas, making them easier to grasp and manipulate
- It reduces ambiguity and errors
- It facilitates teaching, learning, and further research
- Euler's choices were so effective that they became universal, forming the bedrock of modern mathematical language

A universal language: Euler's standardisation of mathematical notation was a gift to all mathematicians, enabling unprecedented clarity and progress

Euler’s dominance in calculus and analysis

Differential calculus
(*Institutiones calculi differentialis*, 1755):

- Provided the first comprehensive and systematic treatment of differential calculus
- Clarified concepts like limits and differentials (though not rigorously by modern standards)
- Developed techniques for higher-order derivatives, implicit differentiation, and the chain rule (if $h = z \circ y$ then $h' = (z' \circ y) \circ y'$)
- Pioneered the use of **partial derivatives** for functions of multiple variables



Euler’s dominance in calculus and analysis

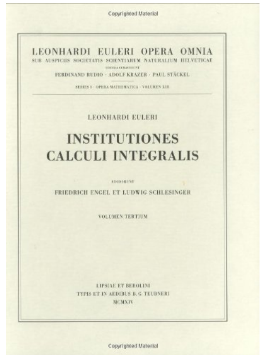
Integral calculus
(*Institutiones calculi integralis*, 1768–1770):

- Provided the first comprehensive textbook on integral calculus.
- Developed standard techniques of integration, including integration by parts and substitution

$$\int fg' = fg - \int f'g$$

$$\int f(x)\,dx = \int f(\phi(t))\phi'(t)\,dt$$

- Introduced and explored various types of integrals, including improper integrals and definite integrals



Euler’s dominance in calculus and analysis

Differential equations:

- Made immense contributions to the theory and solution of **ordinary and partial differential equations**
- Developed methods for solving linear differential equations with constant coefficients
- Applied differential equations to a vast array of problems in physics and mechanics (e.g., fluid flow, celestial mechanics, vibrating strings)

Systematiser and expander: Euler transformed calculus from a collection of powerful techniques into a coherent, organised discipline, ready for its widespread application

Euler and complex numbers

From “impossible” to indispensable:

- Recall Bombelli’s early work with $\sqrt{-1}$ from the Renaissance
- Euler rigorously defined and manipulated complex numbers, proving their immense utility
- Used i for $\sqrt{-1}$, making operations clearer.

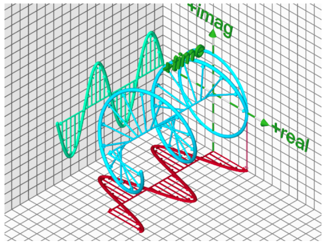
Euler and complex numbers

Euler's formula:

- One of the most beautiful and profound formulas in mathematics:

$$e^{ix} = \cos x + i \sin x$$

- Connects exponential functions (from growth/decay) with trigonometric functions (from circles/oscillations) via the imaginary unit
- It shows that the natural exponential function can be extended to complex exponents via trigonometric functions
- Derived from the power series expansions of e^x, $\sin x$, and $\cos x$



Visualising Euler's formula on the complex plane

Euler and complex numbers

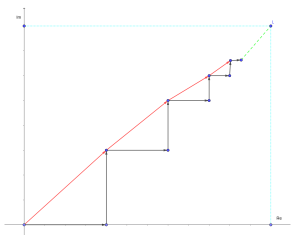
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Then Euler proved

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{i^{2n} x^{2n}}{(2n)!} + \frac{i^{2n} x^{2n+1}}{(2n+1)!} i \right) = \sum_{n=0}^{\infty} \left((-1)^n \frac{x^{2n}}{(2n)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!} i \right) \quad (1) \\ &= \sum_{n=0}^{\infty} \left((-1)^n \frac{x^{2n}}{(2n)!} \right) + \sum_{n=0}^{\infty} \left((-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) i \quad (2) \\ &= \cos(x) + i \sin(x) \end{aligned}$$

Clearly, (1) and (2) are critical

Euler and complex numbers



Step (1) is justified: it is a succession of diagonal arrows in the picture

Step (2) is justified since each series operates on an independent axis

A sound proof but quite far from the contemporary idea of rigour!

Euler and complex numbers

Euler's identity: The "most beautiful equation":

- A special case of Euler's Formula, when $x = \pi$:

$$e^{i\pi} + 1 = 0$$

- Unites five fundamental mathematical constants $(e, i, \pi, 1, 0)$ using the three basic arithmetic operations (addition, multiplication, exponentiation) exactly once
- A testament to the deep interconnectedness of mathematical concepts

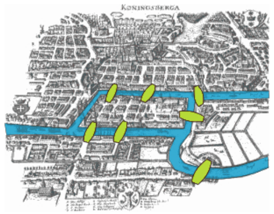
The power of complex analysis: Euler's work with complex numbers opened up the entirely new field of complex analysis, which became vital in physics, engineering, and pure mathematics

Euler's other contributions

Number theory:

- Euler was a monumental figure in reviving and advancing number theory, building on Fermat's work
- **Euler's totient function** ($\phi(n)$): Counts the number of positive integers up to a given integer n that are relatively prime to n. Crucial in cryptography today
- **Euler's criterion**: Used to determine whether an integer is a quadratic residue modulo a prime number
- Proved many of Fermat's theorems, including Fermat's Little Theorem: For every integer a, if p is prime then $a^p \equiv a \pmod{p}$
- Studied partitions of integers

Euler's other contributions



The Königsberg bridge problem

Graph theory:
The Königsberg bridge problem (1735):

- The birth of **graph theory**
- Problem: Can one walk through the city of Königsberg (now Kaliningrad, Russia) and cross each of its seven bridges exactly once?
- Euler proved it was impossible, abstracting the problem into a network of vertices (landmasses) and edges (bridges)
- Introduced the concept of **degree of a vertex** (number of bridges connected to a landmass)
- His solution laid the foundation for an entirely new branch of mathematics

Euler's other contributions

Mechanics and physics:

- Applied his powerful analytical tools to various physical problems
- **Fluid dynamics**: Developed fundamental equations for fluid flow
- **Rigid body dynamics**: Laid the groundwork for their dynamics
- **Lunar theory**: Significant contributions to understanding the moon's complex orbit, crucial for navigation
- **Optics and acoustics**: Also made advances in these fields, often translating physical phenomena into mathematical models

A universal genius: Euler's work not only expanded existing fields but also founded entirely new ones, demonstrating the power of mathematical abstraction and its applicability across diverse domains

The consolidation of calculus

Laying the bedrock of modern mathematical analysis

- **The Enlightenment's push**: The Age of Reason and the rise of Academies created a fertile ground for mathematical expansion
- **The Bernoulli legacy**: A family dynasty that significantly extended and applied calculus, pushing boundaries in probability, series, and physical problems
- **Leonhard Euler**: The colossus of the century:
  - His revolutionary **standardisation of mathematical notation** remains with us today
  - His systematic development of **differential and integral calculus** (including partial derivatives, differential equations) made it a truly powerful and coherent tool
  - His profound work on **complex numbers** (Euler's formula) opened up entirely new fields of analysis
  - His pioneering contributions to **number theory** and the very birth of **graph theory** showcased his universal genius

The consolidation of calculus

Euler and his contemporaries transformed calculus into the universal language for describing change and motion, setting the stage for even deeper analytical inquiries and the development of analytical mechanics in the latter half of the 18^{th} century

References

The main sources for this lecture are

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Dirk J. Struik* A Concise History of Mathematics, Fourth revised edition, Dover Publications (1987)

Also, Euler's depiction in *Eric T. Bell*, Men of Mathematics, Touchstone (1986) is worthwhile

Just to appreciate the amplitude of Euler's work, we strongly suggest searching for his *Opera Omnia*, which is still not concluded

History of Mathematics: Lecture 10

Syllabus:

Mathematics in the 18^{th} century

- Further analytical depths

Beyond Euler

Building on Euler's legacy:

- While Euler dominated the century, other brilliant minds pushed mathematical frontiers
- Focus shifts to deeper theoretical understanding and more complex applications
- Key figures: **Joseph-Louis Lagrange** and **Pierre-Simon Laplace**

Key focus for this lesson:

- The birth of **analytical mechanics**
- Development of the **calculus of variations**
- Refinement of **probability theory**
- Early stirrings of the need for greater mathematical **rigour**

Recap: The age of Euler

Consolidation and explosion of calculus:

- **Context**: The Enlightenment and the rise of scientific academies propelled mathematical research
- **The Bernoulli Family**: Early pioneers who extended calculus, explored probability (law of large numbers), and posed challenging problems (brachistochrone)
- **Leonhard Euler**: The century's colossus
  - His unmatched productivity across all mathematical fields
  - His enduring **standardisation of notation** ($\pi, e, i, f(x), \sum$)
  - His systematic development of **differential and integral calculus**, including differential equations
  - His profound work on **complex numbers**. His formula: $e^{ix} = \cos x + i \sin x$
  - His pioneering work in **number theory** and the very birth of **graph theory**

The previous lesson established calculus as the dominant tool for science. Now, let's see where it leads!

Joseph-Louis Lagrange

Early life and rise to prominence:

- Born in Turin, Italy, of French and Italian descent
- Largely self-taught in mathematics, inspired by Newton and Euler
- Became a professor at the Royal Artillery School in Turin at just 19
- Close collaborator and correspondent with Euler, who recognised his genius
- Succeeded Euler as the director of mathematics at the Berlin Academy (1766-1787)

Joseph-Louis Lagrange (1736-1813)

Joseph-Louis Lagrange

A focus on pure analysis:

- Unlike many of his contemporaries, Lagrange had a strong inclination towards abstract, purely analytical methods
- Sought to express mathematical and physical laws in their most general and elegant analytical form
- His work emphasised algebraic techniques over geometric intuition

Key areas of contribution:

- **Calculus of variations**: A new branch of calculus for optimising functions
- **Analytical mechanics**: Re-founding mechanics purely on analytical principles
- **Number theory**: Significant advancements in the theory of numbers
- Early steps towards **rigour in calculus**

The embodiment of analytical elegance: Lagrange's work epitomised the 18^{th} century's drive for analytical power, systematically deriving results from first principles

Calculus of variations

The "brachistochrone" problem revisited:

- Recall Johann Bernoulli's challenge: finding the curve of fastest descent
- This was one of many problems (e.g., shortest path between points on a surface, minimal surface area) that could not be solved by standard differential calculus (which finds optima of functions, not *paths* or *curves*)
- These problems require optimising a *functional* (a function whose input is another function)

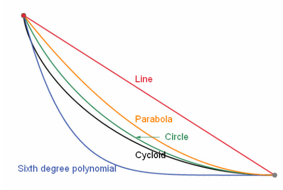
Calculus of variations

Euler and Lagrange's development:

- Euler made significant early contributions to this field
- Lagrange, however, provided a more systematic and general approach
- Developed the **Euler-Lagrange equation**: a differential equation whose solutions are the functions for which a given functional is stationary (i.e., local extrema)

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

where L is the integrand, y is the function, y' its derivative



The brachistochrone problem solution (a cycloid)

Calculus of variations

Impact of calculus of variations:

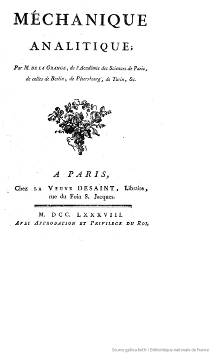
- Essential tool in physics for principles of least action (e.g., Fermat's principle of least time, Hamilton's principle)
- Used in engineering, economics, and optimal control theory
- Showcased the growing power of analytical methods to solve complex optimisation problems that transcended traditional calculus

Optimising functions of functions: The calculus of variations extended the idea of optimisation from single variables to entire functions, leading to profound applications in physics

Analytical mechanics

Mécanique analytique
(Analytical Mechanics, 1788):

- Lagrange's masterpiece, a synthesis of all knowledge of mechanics up to his time
- Its revolutionary aspect: It contains **no diagrams or geometrical constructions**
- All mechanics, from statics to dynamics, is presented entirely through **analytical formulas and algebraic operations**
- Reduced mechanical problems to the solution of differential equations derived from a single principle


MÉCHANIQUE
ANALITIQUE;
Par M. DE LA GRANGE, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.
A PARIS,
Chez LA VEUVE DESAINT, Libraire, rue du Foin S. Jacques.
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Lagrange's *Mécanique analytique*

Analytical mechanics

Lagrangian mechanics:

- Introduced concepts like **generalised coordinates** (any set of independent parameters describing a system's configuration — instead of using a description of the space in which the system operates, he uses a description of the space of states of the system)
- Developed the **Lagrangian** ($L = T - V$, kinetic minus potential energy) and Lagrange's equations of motion
- Provided a powerful, elegant, and unified framework for classical mechanics, simpler (energy-based) for complex systems than the previous Newton's force-based approach

Analytical mechanics

Impact and legacy:

- Transformed mechanics into a branch of pure mathematics, amenable to abstract analysis
- Influenced all subsequent theoretical physics, forming the basis for Hamiltonian mechanics (19^{th} century) and ultimately quantum mechanics and relativity.
- Demonstrated the predictive and explanatory power of mathematics in the natural world

Mechanics as pure algebra: Lagrange's monumental work showcased the power of abstract analysis to describe the physical universe without resorting on geometric crutches

Pierre-Simon Laplace

Pierre-Simon Laplace (1749–1827)

Life and career:

- Born in Normandy, France
- A protégé of d'Alembert (who we'll touch on later)
- Became a leading figure in French science during the tumultuous revolutionary and Napoleonic eras
- Known for his immense intellectual ambition to unify and perfect all knowledge of the universe, particularly through mathematics
- Served in various political roles under Napoleon, though his primary focus remained science

Pierre-Simon Laplace

The systematiser:

- Like Lagrange, Laplace was a master systematiser, aiming to provide comprehensive treatments of entire fields
- Sought to demonstrate the power of analysis in explaining and predicting phenomena in physics and probability

Key areas of contribution:

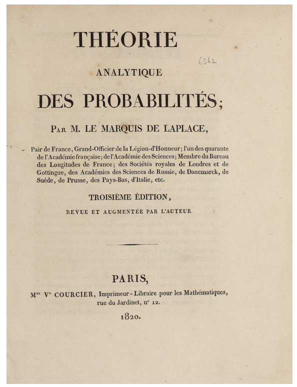
- **Celestial mechanics**: Applying calculus to the solar system's stability and dynamics
- **Probability theory**: Formally establishing its principles and applications
- **Partial differential equations**: Developing tools like the Laplace operator

A synthesis of knowledge: Laplace aimed to create a unified, mathematically consistent model of the universe, demonstrating the ultimate triumph of reason

Probability theory: From games to science

Théorie analytique des probabilités (Analytic Theory of Probability, 1812):

- Laplace's most influential work in probability
- A massive, comprehensive treatise that brought together all previous work and introduced many new ideas
- Consolidated probability theory as a rigorous mathematical science, moving it far beyond its gambling origins



THÉORIE
ANALYTIQUE
DES PROBABILITÉS;
PAR M. LE MARQUIS DE LAPLACE,
Pair de France, Grand-Officier de la Légion-d'Honneur; l'un des quarante de l'Académie française; de l'Académie des Sciences; Membre du Bureau des Longitudes de France; des Sociétés royales de Londres et de Gottingue, des Académies des Sciences de Russie, de Danemarck, de Suède, de Prusse, des Pays-Bas, d'Italie, etc.
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1820.

Laplace's seminal work on probability

Probability theory: From games to science

Key concepts and contributions:

- **Classical definition of probability**: (Number of favourable outcomes) / (Total number of equally likely outcomes)
- **Bayes' theorem**: Though Bayes originated it, Laplace independently rediscovered it and gave it a more general formulation

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- **Central limit theorem** (implicitly): Laid the groundwork for this fundamental theorem of statistics, showing that sums of many independent random variables tend towards a normal distribution
- Applications to statistics, error analysis, population studies, and even jurisprudence

Probability theory: From games to science

The "Philosophical Essay on Probabilities" (1814):

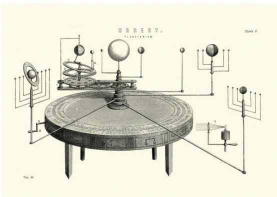
- A popular introduction to his probability theory, explaining the underlying philosophical ideas
- Argued that probability is simply "common sense reduced to calculation".
- Emphasised the predictive power of probability in situations of uncertainty

Quantifying uncertainty: Laplace transformed probability from a collection of isolated problems into a powerful, unified mathematical discipline essential for modern science

Celestial mechanics

Traité de mécanique céleste
(Treatise on Celestial Mechanics, 1799–1825):

- A five-volume magnum opus that applied the full power of calculus (especially differential equations) to the movements of planets, moons, and comets
- Built upon Newton's laws of gravitation but extended them significantly using advanced analytical methods
- Answered long-standing questions about the stability of the solar system, proving that planetary eccentricities and inclinations would remain stable over long periods



Laplace's analytical view of the cosmos

Celestial mechanics

Key mathematical innovations:

- Developed the **Laplace operator** (∇^2) and **Laplace's equation** ($\nabla^2 V = 0$), fundamental in physics (potential theory, electromagnetism) to describe the equilibrium of a system
- Introduced **Laplace transforms**, a powerful tool for solving differential equations (though fully developed later)

Celestial mechanics

Laplace's demon: The ultimate determinism

- A famous philosophical thought experiment from his 1814 essay:

> *We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a given moment knew all the forces that animate nature and the reciprocal positions of the beings that compose it, if this intellect were moreover vast enough to submit these data to analysis, would embrace in a single formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.*

- Embodies the peak of Enlightenment scientific rationalism

Celestial mechanics

A universe as a solvable equation: Laplace's work represented the triumph of analytical methods in describing and predicting the complex workings of the cosmos, solidifying the Newtonian world-machine

Other key figures

Brook Taylor (1685–1731):
The power of series expansions

- English mathematician, contemporary of Newton
- Published *Methodus Incrementorum Directa et Inversa* (1715)
- Introduced **Taylor series** for expanding functions into infinite sums of power terms:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

- A special case, the **Maclaurin series** ($a = 0$), is also widely used
- These series are fundamental for approximating functions and solving differential equations

Other key figures

Jean le Rond d'Alembert (1717–1783)

Jean le Rond d'Alembert (1717–1783):
PDEs and rigour's call

- French mathematician, physicist, philosopher, and co-editor of the *Encyclopédie*
- Pioneering work on **partial differential equations** (PDEs), deriving the one-dimensional **wave equation**: Let $u(x,t)$ be the amplitude in the point x at time t, and let c be the propagation speed of the wave

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Investigated the foundations of calculus, expressing limits using the concept of $\epsilon - \delta$ type arguments (though not fully formalised)
- Questioned the logic of "infinitesimals", contributing to the growing unease about calculus's lack of rigour

Other key figures

Descriptive Geometry (Gaspard Monge, 1746–1818):

- French mathematician, founder of **descriptive geometry**
- Developed methods for representing three-dimensional objects on a two-dimensional plane using projections, crucial for engineering and architecture

Emerging themes

Emerging need for rigour:

- Despite the success of calculus, its foundations were somewhat shaky
- Concepts like "infinitesimals" and "infinity" were used intuitively but lacked precise definitions
- Problems arose with the convergence of infinite series and the behaviour of functions, recall Euler
- Mathematicians like d'Alembert, and later Lagrange, began to voice concerns and make early attempts at more rigorous definitions of limits and continuity
- This intellectual tension would explode with full strength into a major focus in the 19th century

Emerging themes

Setting the stage for the 19th century: The 18th century, while expanding calculus applications, also began to identify foundational gaps, hinting at the need for greater abstraction and rigour

The pinnacle of classical analysis

From calculation to a universal language of science

- **Explosive growth**: The 18th century witnessed an unparalleled expansion and application of calculus, earning it the title "Age of Analysis"
- **Systematisation & notation**: **Euler's** genius unified disparate ideas, standardised notation ($\pi, e, i, f(x), \sum$), and produced comprehensive textbooks, making mathematics accessible and consistent
- **Mastery of mechanics**: **Lagrange's** "Mécanique analytique" transformed classical mechanics into a purely analytical discipline, leveraging the calculus of variations to solve optimisation problems
- **Probability's maturity**: **Laplace** established probability theory as a rigorous science, applying it to vast problems like celestial mechanics and statistical inference

The pinnacle of classical analysis

From calculation to a universal language of science

- **Foundations questioned**: Despite immense success, mathematicians like d'Alembert began to highlight the intuitive (and sometimes imprecise) nature of calculus's foundations, paving the way for the **age of rigour** in the 19th century
- **Interdisciplinary impact**: Mathematics became the indispensable language for physics, astronomy, engineering, and emerging fields like statistics

The 18th century built a robust, powerful analytical edifice, providing the tools and confidence to tackle the grand challenges of science, while simultaneously recognising the need for deeper foundational scrutiny that would define the revolutions of the 19th century

References

The main sources for this lecture are

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Dirk J. Struik* A Concise History of Mathematics, Fourth revised edition, Dover Publications (1987)

For fun, the reader is invited to check the mathematics of "The Orbital Flight Handbook" (1963) in the *Space Flight Handbooks* by NASA to see a practical application of celestial mechanics

History of Mathematics: Lecture 11

Syllabus:

Mathematics in the 19th century:

- The pursuit of rigour and new geometries

Revolutions

The 19th century: A paradigm shift

- A period of unprecedented conceptual revolution in mathematics
- Driven by a demand for **rigour** in analysis (and beyond) and the exploration of abstract structures
- Response to foundational questions raised implicitly in the 18th century (e.g., convergence of series, nature of functions)

Key focus for this lesson:

- The **quest for rigour**: Establishing solid foundations for calculus and its basic notions (Cauchy, Bolzano)
- The revolution in **geometry**: Discovery and development of *non-Euclidean geometries*

Historical context

The industrial revolution's impact on science and technology

The industrial revolution:

- Profound societal, economic, and technological changes
- Creation of new industries (railways, telegraphs, steam power) leading to new engineering and scientific challenges
- Increased demand for precise scientific and mathematical methods

Historical context

Rise of the modern university and research culture:

- Shift from academies (18th century) to research-oriented **universities** as primary centres of mathematical innovation
- **Germany** (especially Göttingen) becomes a dominant force in mathematical research and education
- Emphasis on systematic, rigorous education and original research
- Creation of professional mathematical societies and journals

Historical context

From calculation to conception:

- While the 18th century focused on powerful calculational techniques, the 19th century increasingly emphasised the **foundations** of mathematics
- A shift towards **abstraction** and logical consistency
- Mathematics began to be viewed not just as a tool for science, but as an independent discipline with its own internal logic and beauty

An era of deep inquiry: The 19th century saw mathematics mature into a rigorous, abstract, and internally driven science, profoundly influencing future intellectual thought

Augustin-Louis Cauchy

Life and influence:

- Born in Paris, a brilliant student who thrived in the post-revolutionary French academic system
- A profoundly influential and prolific mathematician, second only to Euler in his output during his lifetime
- Taught at the École Polytechnique, a key institution for engineering and science
- Known for his strict adherence to logical deduction and the need for rigorous proofs
- His work dramatically changed the way mathematics was taught and practised

Augustin-Louis Cauchy (1789–1857)

Augustin-Louis Cauchy

The need for foundations:

- 18th century calculus, though powerful, relied heavily on intuition and geometric reasoning
- Concepts like "limit", "continuity", "infinite series", and similar lacked precise definitions
- This led to paradoxes and incorrect results (e.g., issues with Fourier series)

Key areas of contribution:

- Formalisation of **limits, continuity, and derivatives**
- Rigorous treatment of **infinite series** and their convergence
- Early development of **complex analysis**
- Contributions to differential equations, group theory, and optics

Bringing order to analysis: Cauchy's systematic approach provided the missing logical bedrock for calculus, transforming it into the rigorous subject we know today

Cauchy's definitions

The limit (1821):

- Before Cauchy, a limit was often described loosely as "approaching indefinitely" to some value
- Cauchy provided a precise arithmetical definition, though not explicitly using $\epsilon - \delta$ notation as we do today (that came later, from Weierstrass)
- His definition for a limit of a function $f(x)$ as $x \to a$ is L:

  > *When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit.*

  (Translated from his Cours d'Analyse)
- This moved calculus away from infinitesimals towards a language of vanishing inequalities

Cauchy's definitions

Continuity (1821):

- Defined a continuous function as one where a small change in x produces a small change in $f(x)$
- Formally: A function $f(x)$ is continuous at a if for every quantity ϵ (no matter how small), one can find a quantity δ such that $\left|f(x) - f(a)\right| < \epsilon$ whenever $|x - a| < \delta$
- This precise definition eliminated ambiguity

Cauchy's definitions

Convergence of infinite series (1821):

- Prior to Cauchy, convergence was often assumed or poorly defined
- Cauchy provided the modern definition: an infinite series $\sum a_n$ converges if the sequence of its partial sums $S_n = a_1 + \cdots + a_n$ approaches a finite limit
- Introduced the **Cauchy criterion for convergence**: A sequence converges if and only if its terms become arbitrarily close to each other (for every $\epsilon > 0$, there exists an N such that for all $m, n > N$, $|a_m - a_n| < \epsilon$)
- Sequences satisfying this criterion are now called **Cauchy sequences**. They are fundamental for defining the **completeness** of the real numbers (e.g., in Dedekind's construction)
- This was crucial for avoiding fallacies with series (e.g., those highlighted by Fourier)

Cauchy's definitions

The Cauchy mean value theorem:
If f and g are continuous on $[a,b]$, differentiable in (a,b), and $g'(x) \neq 0$ on (a,b) then there is $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

- A generalisation of Lagrange's mean value theorem, linking the derivatives of two functions over an interval
- Important in the rigorous development of calculus

Building on solid ground: Cauchy's rigorous definitions of fundamental concepts transformed calculus from an art of calculation into a precise science of inequalities

Cauchy and early complex analysis

Building on Euler's foundations:

- Euler had already established complex numbers as valuable tools
- Cauchy, however, began to develop a rigorous theory of functions of a complex variable
- This field, **complex analysis**, became one of the most powerful branches of mathematics, with applications in physics and engineering

Cauchy and early complex analysis

Key concepts:

- **Complex functions**: Functions $f(z)$ where $z = x + iy$
- **Analytic functions**: Functions differentiable in the complex plane
- **Cauchy-Riemann equations**: Conditions that determine if a complex function is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(where $f(z) = u(x,y) + iv(x,y)$)

Cauchy and early complex analysis

- **Cauchy's integral theorem** and **Cauchy's integral formula**: Fundamental results relating integrals of analytic functions along closed paths to their values inside the path. These revolutionised the evaluation of real integrals
- **Integral theorem**: If $f(z)$ is holomorphic in a simply connected domain Ω then for any simply closed contour C in Ω,

$$\int_C f(z)\,dz = 0$$

- **Integral formula**: Let U be an open subset of the complex plane, and suppose the closed disk $D = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ is completely contained in U. Let $f : U \to \mathbb{C}$ be a holomorphic function, and let γ be the circle, oriented counterclockwise, forming the boundary of D. Then for every a in the interior of D

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\,dz$$

Cauchy and early complex analysis

Impact of Cauchy's complex analysis:

- Provided powerful new methods for solving problems in real analysis and applied mathematics
- Laid the groundwork for later developments by Riemann and Weierstrass
- Showed rigour could be applied to complex numbers, not just real ones

A new dimension of analysis: Cauchy transformed complex numbers from a useful mathematical curiosity into a fertile ground for deep and powerful analytical theory

Further steps in rigour

Bernard Bolzano

Bernard Bolzano (1781–1848):
Early insights into continuity

- Bohemian mathematician, philosopher, and theologian
- Independently developed rigorous definitions of **continuity and limits** even before Cauchy, though his work was not widely known until much later

- **Bolzano's theorem** (Intermediate Value Theorem): If a continuous function takes on two values, it must take on every value in between. He provided the first rigorous proof
- Pioneering ideas about **continuous but nowhere differentiable functions**, a concept that challenged intuition and highlighted the need for strict definitions.

Further steps in rigour

The first example of a continuous but nowhere differentiable function is due to Weierstrass

The function was defined as a Fourier series:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \ ,$$

where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + \frac{3}{2}\pi$

Further steps in rigour

Niels Henrik Abel (1802–1829):
Rigour in series and elliptic functions

- Brilliant Norwegian mathematician, tragically short-lived.
- Emphasised the importance of **rigorous proof for series convergence**
- Provided the first rigorous proof of the **binomial theorem** for complex exponents
- **Abel's theorem on binomial series**: Proved its convergence conditions
- Pioneering work on **elliptic functions** (generalising trigonometric functions) and Abelian integrals

Niels Henrik Abel

Further steps in rigour


Peter Gustav Lejeune Dirichlet

Peter Gustav Lejeune Dirichlet (1805–1859): Functions and Fourier series

- German mathematician, a student of Poisson and later taught at Göttingen (succeeding Gauss)
- Provided the modern, formal definition of a **function**: a correspondence that assigns to each element in a set (domain) exactly one element in another set (codomain)
- Made fundamental contributions to the convergence of **Fourier series**, specifying the conditions under which such series converge to the function they represent. This resolved issues faced by earlier mathematicians (e.g., Fourier, Poisson)
- Important work in analytic number theory (e.g., every arithmetic progression $\{a+bn\}_{n\in\mathbb{N}}$ with a and b coprime contains an infinite amount of prime numbers)

Further steps in rigour

Broadening the reach of rigour: These mathematicians built upon Cauchy's foundations, tackling complex analytical problems and further refining the conceptual tools of calculus

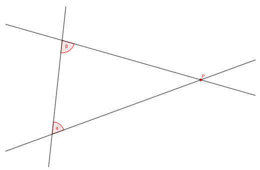
Challenging Euclid's fifth postulate

Euclid's Elements and its fifth postulate:

- For over 2000 years, Euclid's *Elements* stood as the unparalleled model of deductive reasoning
- Its five postulates (axioms) formed the basis of all geometric theorems
- The first four were considered self-evident (e.g., "All right angles are equal to one another")

The **fifth postulate** (parallel postulate) was different:

> *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.*



Challenging Euclid's fifth postulate

A long history of doubts and attempts to prove:

- From ancient Greece (Ptolemy) through the Islamic Golden Age (Omar Khayyam) to the Renaissance (Saccheri, Lambert), mathematicians tried to **prove** the fifth postulate from the first four
- All attempts failed, often leading to bizarre or contradictory conclusions, hinting that it might be independent

Challenging Euclid's fifth postulate

The 19th century breakthrough:

- Instead of proving it, a few brilliant minds began to explore the consequences of **denying** the fifth postulate
- This led to the astonishing discovery of consistent, alternative geometries where Euclid's fifth axiom did not hold
- This revolutionised not only geometry but also the very understanding of axiomatic systems and the nature of mathematical truth

Questioning the obvious: The perceived *flaw* in Euclid's perfect system led to one of the most profound conceptual shifts in mathematical history, opening up unforeseen new worlds of geometry

Carl Friedrich Gauss

Carl Friedrich Gauss

The Prince of Mathematicians:
(1777–1855)

- German mathematician and scientist, considered one of the greatest of all time
- Made fundamental contributions to almost every field of mathematics and science (number theory, algebra, statistics, analysis, differential geometry, geodesy, astronomy, optics, electromagnetism)
- Famously precocious: independently discovered the Fundamental Theorem of Algebra (at age 18) and constructed the 17-gon with ruler and compass (at age 19)

Carl Friedrich Gauss

Early doubts about the fifth postulate:

- As early as the late 1790s, Gauss began to doubt the provability of the parallel postulate
- He secretly developed his own version of **hyperbolic geometry**
- He was convinced of its consistency but never published his findings, fearing "the clamour of the Boeotians" (public misunderstanding)

Carl Friedrich Gauss

Other relevant contributions (brief mention):

- **Differential geometry**: Pioneering work on the intrinsic geometry of surfaces (*Theorema Egregium*). This laid groundwork for Riemann
- **Complex numbers**: Showed they were a natural extension of real numbers, and used them geometrically
- **Least squares**: Fundamental in statistics and error theory

A secret revolutionary: Gauss's unshared insights into non-Euclidean geometry underscore his visionary genius and the profound paradigm shift occurring in mathematics

Lobachevsky and Bolyai

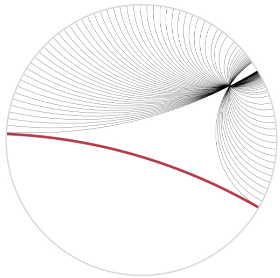
Nikolai Lobachevsky (1792–1856): Russian maverick

- Professor at Kazan University.
- Published his work on "imaginary geometry" in 1829.
- He allowed that through a point not on a given line, there can be **more than one parallel line** to the given line
- Developed the consistent logical consequences of this assumption, creating what we now call **hyperbolic geometry**

János Bolyai (1802–1860): Hungarian pioneer

- Son of Farkas Bolyai, a friend of Gauss
- Independently developed similar ideas around the same time
- Published his work as an appendix to his father's textbook in 1832, calling it "The absolute science of space"
- His father shared his son's work with Gauss, who famously replied: "To praise it would be to praise myself"

Hyperbolic geometry



Visualising hyperbolic geometry: Poincaré disk model

The nature of hyperbolic geometry:

- **Key difference**: Through a point not on a line, infinitely many lines can be drawn parallel to the given line
- Triangles in hyperbolic space have the sum of their angles **less than 180 degrees**
- The geometry of a saddle surface or hyperbolic paraboloid
- Initially met with scepticism, but later understood as equally consistent as Euclidean geometry

A bold departure: Lobachevsky and Bolyai's independent publications shattered the belief that Euclidean geometry was the only possible geometry, opening the door to revolutionary new ways of understanding space

Bernhard Riemann

A visionary student of Gauss:

- German mathematician, one of Gauss's last and most brilliant students
- Known for his profound and highly abstract contributions to geometry, complex analysis, and number theory
- His 1854 habilitation lecture, *On the Hypotheses which Lie at the Bases of Geometry*, revolutionised the concept of space itself

Bernhard Riemann (1826–1866)

Bernhard Riemann

Riemannian geometry: The geometry of curved manifolds

- Instead of just negating a postulate, Riemann generalised geometry to allow for spaces with **variable curvature** at every point
- Introduced the concept of an ***n*-dimensional manifold**: a space that locally resembles Euclidean space but can be globally curved
- Defined the **Riemannian metric**: a way to measure distances and angles on these curved spaces, using differential calculus

Bernhard Riemann

Elliptic geometry (A special case):

- In this geometry, through a point not on a given line, there are **no parallel lines**
- The sum of angles in a triangle is **greater than 180 degrees**
- Can be visualised as the geometry on the surface of a sphere (e.g., lines are great circles, triangles formed by meridians and equator)
- Has constant positive curvature

Impact on physics of 20th century: Riemannian geometry provided the mathematical framework for **Einstein's theory of general relativity**, where gravity is understood as the curvature of spacetime itself

Foundations rebuilt, universes reimagined

A new era of rigour, abstraction, and conceptual freedom

- **The quest for rigour**: Driven by the need to resolve paradoxes and establish certainty, mathematicians like **Cauchy** provided the precise, arithmetical definitions of limits, continuity, and convergence that underpin modern analysis. **Bolzano**, **Abel**, and **Dirichlet** further refined these foundations
- **Revolutionary geometries**: The long-standing parallel postulate was finally understood as independent, leading to the astonishing discoveries of consistent **non-Euclidean geometries** by **Lobachevsky**, **Bolyai**, and implicitly **Gauss**. **Riemann**'s work generalised geometry to curved manifolds, laying the groundwork for modern physics

Foundations rebuilt, universes reimagined

A new era of rigour, abstraction, and conceptual freedom *The first half of the 19th century shattered centuries-old assumptions about numbers, space, and algebraic structures, replacing them with a newfound rigour and a powerful embrace of abstract thinking. This intellectual freedom opened doors to even more profound and surprising discoveries yet to come*

References

The main sources for this lecture are

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)

(H.P. Lovecraft, *The Call of Cthulhu*, 1926)

Without knowing what futurism is like, Johansen achieved something very close to it when he spoke of the city; for instead of describing any definite structure or building, he dwells only on broad impressions of vast angles and stone surfaces — surfaces too great to belong to any thing right or proper for this earth, and impious with horrible images and hieroglyphs. I mention his talk about angles because it suggests something Wilcox had told me of his awful dreams. He had said that the geometry of the dream-place he saw was abnormal, non-Euclidean, and loathsomely redolent of spheres and dimensions apart from ours. Now an unlettered seaman felt the same thing whilst gazing at the terrible reality.

History of Mathematics: Lecture 12

Syllabus:

Mathematics in the 19th century:

- Deepening analysis and new abstractions

Deepening analysis and new abstractions

Building on a century of revolution:

- Recall previous lesson: The seismic shifts in foundations (rigour) and geometry (non-Euclidean)
- This lesson continues the trend of abstraction and deeper inquiry
- Mathematicians now had a clearer understanding of what **proof** meant

Key focus for this lesson:

- The **arithmetisation of analysis**: Establishing real numbers and functions on a purely numerical basis (Weierstrass, Dedekind)
- The expansion of **abstract algebra**: From groups (Galois) to new algebraic structures (Hamilton, Cayley, Grassmann)
- The revolutionary world of **set theory**: Confronting the deep nature of infinity (Cantor)

Foundations rebuilt, universes reimagined

Rigour, non-Euclidean worlds, and group symmetries

- **The quest for rigour**: **Cauchy** provided precise, arithmetical definitions for limits, continuity, and convergence, laying the modern foundation for analysis. Figures like **Bolzano**, **Abel**, and **Dirichlet** further contributed
- **Revolutionary geometries**: Challenging Euclid's fifth postulate led to the discovery of consistent **non-Euclidean geometries** (e.g., hyperbolic geometry by **Lobachevsky** and **Bolyai**, with **Gauss**'s earlier, unshared insights). **Riemann** generalised geometry to curved manifolds, foreshadowing relativity

Previous lesson revealed that mathematics was not static but a dynamic field capable of fundamental conceptual shifts.
Now, let's explore how these shifts deepened!

Karl Weierstrass

From high school teacher to "Father of Modern Analysis":

- German mathematician, initially studied law and finance, later mathematics
- Spent years as a gymnasium (high school) teacher before his extraordinary research was recognised, leading to a professorship in Berlin
- Renowned for his meticulous and systematic approach, and his insistence on absolute rigour
- His influential teaching at Berlin shaped a generation of mathematicians

Karl Weierstrass (1815–1897)

Karl Weierstrass

Beyond Cauchy: The $\epsilon - \delta$ formalism:

- While Cauchy began the process of rigour, his definitions still had some ambiguities (e.g., in the distinction between pointwise

$$\forall \varepsilon > 0, x \in E\, \exists N \in \mathbb{N}\, \forall n > N \left| f_n(n) - f(x) \right| < \varepsilon$$

  and uniform convergence

$$\lim_{n\to\infty} \left(\sup_{x \in E} \left| f_n(x) - f(x) \right| \right) = 0$$

  of a sequence $\{f_n\}_{n\in\mathbb{N}}$ to f on E)
- Weierstrass provided the definitive, completely arithmetical definitions of fundamental concepts
- He fully formalised the **$\epsilon - \delta$ definition of a limit** and **continuity**:
  - $L = \lim_{x\to a} f(x)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$
- This precise language eliminated reliance on geometric intuition

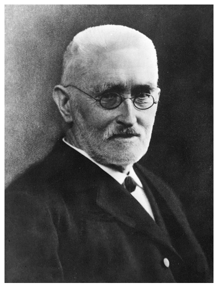
Karl Weierstrass

Key contributions to rigorous analysis:

- **Uniform convergence**: Crucial for interchanging limit operations (e.g., integral of a limit, derivative of a series)
- **Continuous, nowhere differentiable functions**: Constructed the first formal example (1872) of a function that is continuous everywhere but differentiable nowhere, challenging classical intuition
- **Foundation of real numbers**: His program aimed to base all of analysis on properties of integers, leading to Dedekind's work
- **Analytic functions through power series**: Believed that analytic functions (representable by power series) should be the fundamental objects of study in analysis

The pinnacle of rigour: Weierstrass completed the arithmetisation of analysis, establishing the rigorous framework that continues to define mathematical analysis today

Richard Dedekind



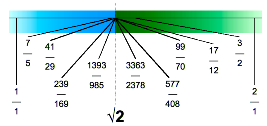
Richard Dedekind (1831–1916)

The missing foundation:

- Even with Cauchy's and Weierstrass's rigour for limits and continuity, the **real numbers** themselves lacked a precise, logical foundation
- What exactly **is** an irrational number? How do we define continuity of the real line?
- This foundational gap needed to be addressed to truly "arithmetise" analysis

Richard Dedekind

Dedekind cuts (1872):



- Dedekind, a student of Gauss, provided a rigorous construction of the real numbers from the rational numbers
  - A **Dedekind cut** is a partition of the rational numbers $\mathbb{Q}$ into two non-empty sets, A and B, such that:
    1. Every rational number is in either A or B
    2. Every element of A is less than every element of B
    3. A contains no greatest element
  - This cut uniquely defines a real number (rational if B has a least element, irrational otherwise)
  - This concept precisely captured the "completeness" of the real number line, ensuring there are "no gaps"

Richard Dedekind

Impact of Dedekind's work:

- Provided the logical bedrock for all of analysis by rigorously defining the numbers used
- Showed that the continuum of real numbers could be constructed purely from the simpler rational numbers
- Cemented the arithmetisation program by placing analysis on an unshakeable numerical foundation

Defining the continuum: Dedekind's construction of the real numbers removed the last major intuitive ambiguity from the foundations of analysis, completing its arithmetisation

Évariste Galois

Évariste Galois (1811–1832)

A brief, tragic life:

- French mathematician whose genius was largely unrecognised during his lifetime
- A republican activist during a turbulent political period in France, leading to arrests and imprisonment
- Died in a duel at the age of 20, leaving behind revolutionary mathematical manuscripts
- His work was only fully understood and published posthumously (by Liouville in 1846)

Évariste Galois

Solvability of polynomial equations:

- For centuries, mathematicians sought formulas to solve polynomial equations using radicals (square roots, cube roots, etc.)
- Quadratic formula (2^{nd} degree): $ax^2 + bx + c = 0$
- Cubic (3^{rd} degree) and quartic (4^{th} degree) formulas were found in the 16^{th} century (Cardano, Tartaglia, Ferrari)
- A long-standing quest was to find a general formula for the **quintic equation** (5^{th} degree) and higher
- **Abel-Ruffini theorem** (Ruffini, 1799; Abel, 1824): Proved that no general formula exists for solving quintic or higher-degree polynomial equations using only radicals
  For example $x^5 - x + 1 = 0$ has four complex roots and a real one, which cannot be expressed using radicals

Évariste Galois

Galois's revolutionary insight: Symmetry and groups

- Instead of looking for a formula, Galois asked: *Under what conditions* can a polynomial equation be solved by radicals?
- His answer lay in the **symmetries of the roots** of the equation
- He invented **group theory** to describe these symmetries. A group is a set with a binary operation satisfying certain axioms (closure, associativity, identity, inverse)
- He showed that a polynomial equation is solvable by radicals if and only if its associated **Galois group** has a specific structure (being "solvable")

The dawn of abstract algebra:
Galois's work introduced abstract algebraic structures (groups) to solve a concrete problem, opening up an entirely new way of doing mathematics based on the study of abstract systems and their symmetries

William Rowan Hamilton

The quest for numbers in 3D:

- Irish mathematician, physicist, and astronomer. A child prodigy, fluent in many languages
- Professor of Astronomy at Trinity College Dublin at 22, while still an undergraduate
- Fascinated by extending complex numbers (which model 2D rotations and scaling) to represent points or transformations in 3D space
- Spent years trying to define a consistent "multiplication" for triplets of numbers ($a + bi + cj$) that behaved like standard numbers

William Rowan Hamilton (1805–1865)

William Rowan Hamilton

The eureka moment (1843):
The non-commutative *quaternions*:

- While walking with his wife along Dublin's Royal Canal, the solution suddenly struck him
  He carved the famous relations onto Brougham Bridge:

$$i^2 = j^2 = k^2 = ijk = -1$$

- This led to the discovery of **quaternions**, a number system of the form $a + bi + cj + dk$, where a, b, c, d are real numbers, and i, j, k are imaginary units (roots of $\sqrt{-1}$)
- The crucial insight: multiplication for quaternions is **non-commutative** ($ij = k$ but $ji = -k$)

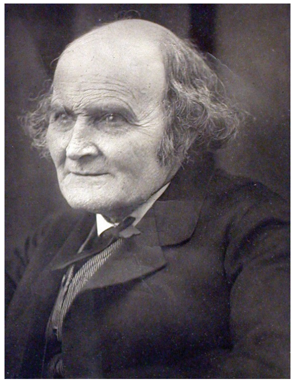
William Rowan Hamilton

Impact and legacy:

- A radical departure from traditional arithmetic, where multiplication was always assumed to be commutative
- Demonstrated that new, consistent algebraic systems could be created by **relaxing** an axiom (commutativity in this case)
- Initially seen as a powerful tool for 3D rotations in physics, though later largely superseded by vector algebra (developed by Gibbs and Heaviside from Grassmann's ideas)
- Fundamental for the development of modern **abstract algebra**, particularly the study of non-commutative rings and algebras
- Still used in computer graphics, aerospace engineering (for orientation), and quantum mechanics

Algebra's new freedom: Hamilton's quaternions broke the shackles of arithmetic, revealing that algebra could explore entirely new structures, not just generalise existing ones

Arthur Cayley



Arthur Cayley (1821–1895)

English polymath and legal professional:

- English mathematician, one of the most prolific of the 19th century, publishing over 900 papers
- Held a prestigious fellowship at Trinity College, Cambridge, but also pursued a career as a lawyer for 14 years, practising conveyancing
- His legal work provided him with financial independence, allowing him to pursue mathematics without academic pressures
- Later became the first Sadlerian Professor of Pure Mathematics at Cambridge

Arthur Cayley

The birth of matrix theory (1850s):

- While matrices appeared implicitly earlier (e.g., in Gauss's work on quadratic forms), **Cayley was the first to develop the algebra of matrices explicitly**
- Defined matrix addition, scalar multiplication, and most importantly, **matrix multiplication** (which is also non-commutative, like Hamilton's quaternions, but in a different context)
- Realised that matrices could represent **linear transformations** (rotations, scaling, shears) in a concise and general way

Arthur Cayley

- **Cayley-Hamilton theorem**: Every square matrix satisfies its own characteristic polynomial: let A be a $n \times n$ matrix and I be the n-identity matrix, then

$$\det(A - \lambda I) = \sum_{i=0}^{n} c_i \lambda^i$$

with $c_n = 1$. Posing $A^0 = I$ and 0_n the zero square matrix $n \times n$, the theorem tells that

$$\sum_{i=0}^{n} c_i A^i = 0_n$$

(proved by Cayley for 2×2 and 3×3, later generalised)

Arthur Cayley

Broader impact on abstract algebra:

- Laid the foundation for **linear algebra** as an abstract subject, independent of specific coordinates or geometric interpretations
- Contributed to the development of **group theory**, providing concrete examples of groups as collections of permutations (Cayley's Theorem states every group is isomorphic to a group of permutations)
- Pioneering work on **invariant theory**, studying properties that remain unchanged under transformations

The language of transformations: Cayley's abstract formulation of matrices provided a powerful new language for studying linear transformations and ushered in the era of modern linear algebra, foundational to physics, computer science, and engineering

Hermann Grassmann

A visionary ahead of his time:

- German linguist, physicist, and mathematician
- Primarily a high school teacher (gymnasium), his radical mathematical ideas were initially largely ignored or misunderstood by his contemporaries
- His work was far too abstract for its time, only gaining significant recognition in the late 19^{th} and early 20^{th} centuries

Hermann Grassmann (1809–1877)

Hermann Grassmann

The *Ausdehnungslehre* (Theory of Extension, 1844/1862):

- His most significant work, attempting to create a universal algebraic language for all of geometry and physics
- Fundamental concepts to linear algebra and differential geometry:
  - **Vector spaces**: Abstract spaces where elements (vectors) can be added and scaled, without reliance on coordinates
  - **Linear independence** and **dimension**
  - **Inner product** (dot product) for measuring angles and lengths: $\langle \_ , \_ \rangle : V \times V \to \mathbb{R}\ (\mathbb{C})$ such that $\langle v, v \rangle \geq 0$ with equality holding if and only if $v = 0$, and $\langle u, v \rangle = \langle v, u \rangle\ (\overline{\langle v, u \rangle})$
- Developed **exterior algebra** (or Grassmann algebra): A system for manipulating higher-dimensional quantities like areas (2-vectors) and volumes (3-vectors) in an abstract, coordinate-free way. This involves the "wedge product" ($\wedge$)

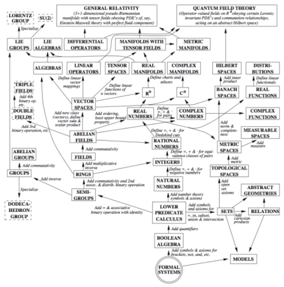
Hermann Grassmann

Impact and legacy:

- His ideas were a crucial conceptual precursor to modern vector calculus and tensor analysis
- Provided the abstract framework that underpins much of 20^{th} century physics (e.g., electromagnetism, general relativity) and computer graphics
- Demonstrated that geometry could be studied algebraically, not just visually or axiomatically
- Emphasised the power of abstract definition, allowing for structures far removed from physical intuition

The foundation of modern geometry and physics: Grassmann's highly abstract theories provided the conceptual tools to describe geometric and physical quantities in any number of dimensions, paving the way for manifold theory and modern physics

The universe of abstract algebraic structures



The hierarchy of algebraic structures

From concrete to abstract:

- Starting with **Galois's groups** (symmetries of roots)
- Moving to **Hamilton's quaternions** (non-commutative numbers)
- And **Cayley's matrices** (non-commutative transformations)
- Further abstract notions of **vector spaces** by Grassmann (sets with addition and scalar multiplication)

The 19^{th} century saw a gradual but definitive shift from studying **specific** numbers or operations to defining **general sets with operations** that satisfy certain axioms

The universe of abstract algebraic structures

Emergence of other structures:

- While formal definitions came later, the groundwork for other fundamental algebraic structures was laid
  - **Rings**: Sets with two operations (like addition and multiplication) that satisfy certain properties (e.g., integers, polynomials). The concept was implicit in number theory (Dedekind, Kronecker)
  - **Fields**: Rings where every non-zero element has a multiplicative inverse (e.g., rational numbers, real numbers, complex numbers). Important for Galois theory
- The focus moved to understanding the **properties** of these abstract systems rather than just calculating with specific numbers

The universe of abstract algebraic structures

Why this abstraction? Theoretical power:

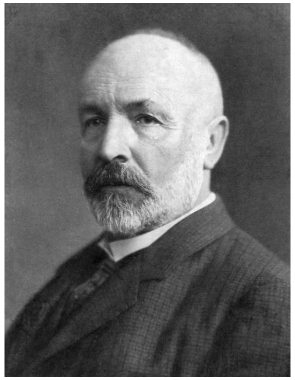
- **Generalisation**: Theorems proven for an abstract group (or ring, or vector space) apply to **all** concrete examples of that structure
- **Unified framework**: Different areas of mathematics (number theory, geometry, analysis) could be seen to share underlying algebraic structures
- **New problems**: The study of these structures became a rich field of research, leading to deep questions about classification and properties

From solving equations to building worlds: The 19^{th} century transformed algebra from the science of equations into the science of abstract structures, providing a universal language for patterns and symmetries across mathematics

Georg Cantor

A pioneer of the infinite:

- German mathematician, born in Russia. Studied at Zürich and Berlin (under Weierstrass)
- Began his research trying to understand the convergence of Fourier series, which led him to questions about the nature of point sets
- His work on infinity was highly original, deeply philosophical, and initially met with strong resistance and controversy
- He faced significant personal struggles and professional opposition, notably from Leopold Kronecker



Georg Cantor (1845–1918)

Georg Cantor

The problem: What is *infinity*?

- Ancient Greek mathematicians (e.g., Zeno's paradoxes) avoided the actual infinite, dealing only with "potential infinity"
- 18^{th} century calculus used infinity informally (e.g., infinite series, limits approaching infinity)
- Cauchy and Weierstrass formalised limits using finite quantities, but the nature of infinite **sets** remained vague
- Cantor dared to ask: Can there be **different sizes** of infinity?

Georg Cantor

Key concepts introduced:

- **Set theory**: The fundamental theory of collections of objects
- **One-to-one correspondence**: A method for comparing the "size" of sets (cardinality), even infinite ones
- **Cardinal numbers**: A way to quantify the size of sets (e.g., $\aleph_0$)
- **Transfinite numbers**: Numbers representing different levels of infinity

Redefining the foundations of mathematics: Cantor's set theory provided a new language and framework for all of mathematics, forcing a re-evaluation of fundamental concepts like number, quantity, and space

Different infinities!

Countable infinity ($\aleph_0$):

- A set is **countably infinite** if its elements can be put into a one-to-one correspondence with the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
- Examples:
  - The set of **integers** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (can be "counted" as $0, 1, -1, 2, -2, \ldots$)
  - The set of **rational numbers** $\mathbb{Q}$ (Cantor showed this is also countable using a diagonalisation argument, pairing numerators and denominators)
- The cardinality of these sets is denoted by $\aleph_0$ (aleph-null or aleph-zero)

Different infinities!

Uncountable infinity ($\mathfrak{c}$):

- Cantor's groundbreaking discovery: Some infinite sets are "larger" than others
- The set of **real numbers** $\mathbb{R}$ is **uncountably infinite**
- **Cantor's diagonalisation argument** (1874): Proved that for any assumed list of real numbers between 0 and 1, one can always construct a new real number not on the list
- This demonstrated that the continuum of real numbers cannot be put into one-to-one correspondence with the natural numbers

Different infinities!

The continuum hypothesis:

- Cantor hypothesised that there is no set whose cardinality is strictly between that of the natural numbers ($\aleph_0$) and the real numbers ($\mathfrak{c}$)
- That is, $\mathfrak{c} = \aleph_1$ (the next cardinal number after $\aleph_0$)
- This problem remained unsolved for over a century and was later shown to be **independent** of the standard axioms of set theory (Gödel 1940, Cohen 1963)

Different infinities!

Cantor's theorem (power set):

- For any set A, the power set $\mathscr{P}(A)$ (the set of all subsets of A) has a strictly greater cardinality than A itself
- This implies an infinite hierarchy of infinities: $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \ldots$

A universe of infinities: Cantor's work unveiled a rich and complex structure within the realm of the infinite, forever changing our understanding of quantity and laying the groundwork for mathematical logic

Towards a new mathematical universe

From arithmetisation to the abstract and the infinite

- **The arithmetisation of analysis**: The foundations of calculus were fully solidified by **Weierstrass**'s rigorous $\epsilon - \delta$ definitions and **Dedekind**'s construction of the real numbers, removing all remaining ambiguities and intuitive leaps
- **Expanding abstract algebra**: The bold steps taken by **Hamilton** (quaternions), **Cayley** (matrices), and **Grassmann** (vector spaces, multilinear algebra) demonstrated the power and freedom of defining abstract algebraic structures beyond traditional arithmetic
- **The revolution of set theory: Cantor**'s groundbreaking work unveiled the hierarchy of different infinities (**countable** vs. **uncountable**), laying the groundwork for a new fundamental theory but also stirring controversy

Towards a new mathematical universe

The 19th century transformed mathematics from a science of quantity and calculation into a rigorous, abstract discipline centred on the study of structures, patterns, and the very nature of mathematical objects. This unprecedented level of abstraction, while powerful, also led to unforeseen challenges and paradoxes, setting the stage for a dramatic foundational re-evaluation at the turn of the 20th century

References

The main sources for this lecture are

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)

For the fun of the reader, we suggest reading any puzzle book by *Raymond Smullyan*, for example, "Satan, Cantor and Infinity" (1992): they are a playful but still rigorous introduction to the mysteries of set theory

History of Mathematics: Lecture 13

Syllabus:

The turn of the 19th and 20th Century:

- Foundations in crisis, logic reborn

Foundations in crisis, logic reborn

A period of profound transition:

- Following the revolutionary 19th century, mathematics reached new heights of abstraction and rigour
- This period, roughly 1890–1930, was marked by both a synthesis of prior achievements and a critical examination of mathematical foundations
- Question: Has mathematics achieved absolute certainty, or are there cracks in the bedrock?

Key focus for this lesson:

- **Foundational crises**: The emergence of paradoxes in set theory (Russell's Paradox)
- **The program for mathematics**: Hilbert's problems and the quest for the consistency of Mathematics
- **Birth of mathematical logic**: Attempts to formalise and axiomatise mathematics (Frege, Peano, Russell, Whitehead)
- **New fields emerge**: Measure theory (Lebesgue) and topology (Poincaré)

The 19th century's legacy

Rigour, abstraction, and the untamed infinite

- **Rigorous analysis**: **Weierstrass** and **Dedekind** built calculus on solid arithmetical foundations, eliminating intuition. But did this make mathematics **absolutely** safe?
- **Abstract algebra**: **Galois**, **Hamilton**, **Cayley**, and **Grassmann** showed the power of abstract structures (groups, quaternions, matrices, vector spaces). This led to immense generalisation but also questions: What **is** a mathematical object?
- **Set theory and the infinite**: **Cantor**'s groundbreaking work revealed different sizes of infinity and made the infinite an **actual**, manipulable mathematical object. This was revolutionary but also deeply unsettling and led to its own internal difficulties

The very tools that brought unprecedented power and clarity to 19th century mathematics also revealed its latent vulnerabilities, particularly in the realm of set theory, sparking a profound crisis of confidence

Unsettling paradoxes in set theory

The unforeseen consequence of abstraction:

- The 19th century's drive for rigour and abstraction, especially Cantor's set theory, led to unprecedented conceptual power
- However, it also exposed hidden vulnerabilities in the underlying assumptions about sets and logic
- A *paradox* in mathematics is not just a tricky problem, but a seemingly valid deduction from accepted axioms that leads to a contradiction
- Such contradictions threatened the very consistency and reliability of mathematics

Naive set theory's premise:

- Cantor's implicit assumption (and initially, that of others):
  Any "well-defined collection of objects" can form a set
- This seemingly intuitive idea was the fertile ground for paradoxes

Unsettling paradoxes in set theory

Key paradoxes emerged:

- **Burali-Forti paradox** (1897): Concerns the set of all ordinal numbers. Its existence leads to a contradiction
- **Cantor's paradox** (1899): Concerns the "set of all sets". If such a set exists, its power set must have a greater cardinality, yet it must already contain all sets
- **Russell's paradox** (1901): Perhaps the most famous and impactful, demonstrating a flaw in the very definition of a set

A jolt to certainty:
These paradoxes revealed that even seemingly simple and intuitive concepts, when pushed to their logical limits, could lead to fundamental inconsistencies, demanding a re-evaluation of mathematical foundations

Russell's paradox

The simple, devastating contradiction:

- Formulated by **Bertrand Russell** (1901)
- It exposed a fatal flaw in naive set theory's unrestricted *comprehension axiom* (that any well-defined property defines a set)
- Consider the set R of all sets that are **not members of themselves**
  - $R = \{x \mid x \notin x\}$
- Now, ask the question: **Is R a member of itself ($R \in R$)**?
  - If $R \in R$, then by definition of R, $R \notin R$. (Contradiction!)
  - If $R \notin R$, then by definition of R, $R \in R$. (Contradiction!)
- This leads to a logical contradiction, meaning the set R cannot exist under the naive assumption

Russell's paradox

Analogy: the barber paradox

- In a village, there is a barber who shaves all and only those men in the village who do not shave themselves
- Who shaves the barber?
  - If the barber shaves himself, then by definition, he does not shave himself
  - If the barber does not shave himself, then by definition, he shaves himself
- This simple analogy captures the essence of Russell's paradox

The impact: Devastating to foundational programs Russell's paradox immediately undermined the nascent logicist program of **Gottlob Frege**, who was attempting to derive all of mathematics from logic and naive set theory. It showed that fundamental assumptions were flawed

Competing philosophies

Three main schools of thought emerged:

- **Logicism:** Aimed to reduce all of mathematics to logic. Mathematics is seen as a branch of logic
  - **Proponents**: Gottlob Frege, Bertrand Russell, Alfred North Whitehead
  - **Challenge**: Undermined by Russell's Paradox (as discussed)
- **Intuitionism**: Argued that mathematics is fundamentally a mental construction, and only constructively proven objects exist. Rejected the Law of Excluded Middle for infinite sets
  - **Proponents**: L.E.J. Brouwer
  - **Challenge**: Very restrictive paradigm, a large part of classical mathematics deemed non-existent
- **Formalism**: Viewed mathematics as a formal game played with symbols according to strict rules (axioms). The goal was to prove the consistency of these axiomatic systems
  - **Proponent**: David Hilbert.
  - **Promise**: Offered a path to absolute certainty

David Hilbert

David Hilbert (1862–1943)

The program for mathematics

- German mathematician, one of the most influential of the 20th century
- Believed mathematics could be placed on a firm, axiomatic foundation
- His "program" aimed to:
  - Prove the **consistency** of arithmetic using finite, indisputable methods (finitary methods)
  - Show the **completeness** of axiomatic systems (all true statements are provable)
  - Ensure the **decidability** of mathematical statements (an algorithm exists to determine truth/falsity)

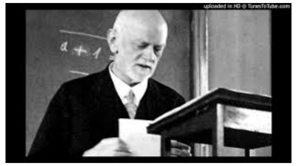
David Hilbert

A grand vision for certainty: Hilbert's program was the leading attempt to establish a definitive, unshakeable foundation for all of mathematics, driven by the desire to overcome the foundational crises

Hilbert's problems

The Paris address:

- At the **International Congress of Mathematicians in Paris in 1900**, David Hilbert delivered a seminal address
- He presented a list of **23 unsolved mathematical problems** that he believed would be central to the development of mathematics in the coming century
- This list became an immense driving force for research, shaping the direction of mathematics for decades
- Many problems have since been solved, some partially, and a few remain open



Hilbert's 23 problems for the 20^{th} century

Hilbert's problems

Problems related to foundations:

- **Problem 1: The continuum hypothesis**: Is there a set whose cardinality is strictly between that of the natural numbers and the real numbers? (Cantor's problem)
- **Problem 2: The consistency of arithmetic**: Is the system of axioms for elementary arithmetic free from contradiction? Can this consistency be proven using only finitary means? (central to Hilbert's program)
- **Problem 10: Diophantine equations**: Given a Diophantine equation with any number of unknown quantities and with rational integral coefficients, to devise a process by which it can be determined in a finite number of operations whether the equation has a solution in rational integers. (The "decidability" aspect, later shown to be impossible by Matiyasevich's theorem)

David Hilbert

The axiomatic method:

- Hilbert championed the **axiomatic method**: defining mathematical structures (like geometry or numbers) through a small set of basic assumptions (axioms) and logically deriving all theorems
- This approach aimed to bring rigour and clarity to every branch of mathematics, with no appeal to interpretation
- Example: His work on the foundations of geometry, showing its consistency if arithmetic is consistent

A blueprint for progress and a battle for truth: Hilbert's problems not only directed research but also formalised the grand challenge of the early 20^{th} century: to establish the absolute certainty and self-consistency of mathematics

Towards mathematical logic

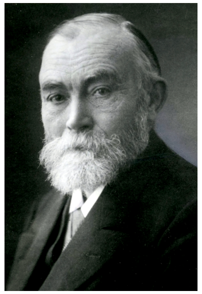
The roots of symbolic logic:

- The formal, symbolic approach to logic began earlier with figures like **George Boole** (mid-19th century), who developed Boolean algebra
- This provided an algebraic system for logical operations (AND, OR, NOT)
- However, to formalise mathematics itself and its internal structure, a more expressive system (like predicate logic) was needed, leading to Frege's work

Gottlob Frege

A vision of logic as the foundation of mathematics:

- German mathematician, logician, and philosopher
- Largely ignored during his lifetime, his work was tragically undermined by Russell's paradox
- His ultimate goal (the **logicist program**) was to demonstrate that **arithmetic could be derived entirely from logic**, without recourse to intuition or empirical observation



Gottlob Frege (1848–1925)

Gottlob Frege

Key innovations in formal logic:

- **Begriffsschrift** (Concept-script, 1879): His groundbreaking work, arguably the most important single work in logic after Aristotle
- Invented **predicate logic** (or quantifier logic):
  - Introduced **quantifiers** ($\forall$ "for all", $\exists$ "there exists") to express generality and existence precisely
  - Distinguished between concepts (predicates) and objects, moving beyond the simple subject-predicate structure of traditional Aristotelian logic
- Developed a rigorous **formal symbolic language** for logic, allowing for unambiguous and mechanical derivation of proofs
- Pioneered the idea of a **formal system** with axioms and rules of inference

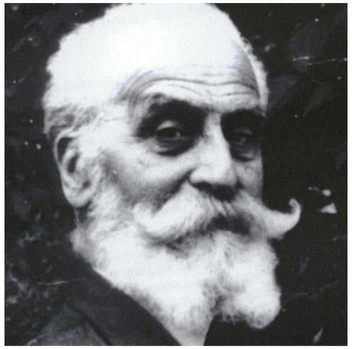
Gottlob Frege

Grundgesetze der Arithmetik (Basic Laws of Arithmetic, 1893, 1903):

- His ambitious attempt to execute the logicist program
- In volume I (1893), he set out axioms from which he believed all of arithmetic could be derived
- Just as volume II (1903) was going to press, he received a letter from **Bertrand Russell**, pointing out the devastating paradox in his system (Russell's Paradox)

A masterpiece undermined: Frege's work revolutionised logic, but his grand project to derive mathematics from it was tragically incomplete due to unforeseen paradoxes, requiring future mathematicians to rebuild the foundations of set theory

Giuseppe Peano



Giuseppe Peano (1858–1932)

Italian mathematician and pioneer of axiomatics:

- Italian mathematician, logician, and linguist
- Focused on rigorously formalising existing mathematical theories, particularly arithmetic
- Advocated for the clear and precise use of symbolic language in mathematics

Giuseppe Peano

Peano axioms (1889):

- A set of five axioms that formally define the natural numbers ($\mathbb{N}$) and their properties (addition, multiplication)
- These axioms are the standard foundation for arithmetic today
  1. 0 is not the successor of any natural number
  2. If two natural numbers have the same successor, then they are equal
  3. The inductive definition of addition
  4. The inductive definition of multiplication
  5. (Principle of mathematical induction): If a property holds for 0, and holds for the successor of any number for which it holds, then it holds for all natural numbers
- This provided a concrete, consistent axiomatic basis for the most fundamental numbers

Giuseppe Peano

Contributions to symbolic logic and notation:

- Developed much of the notation that became standard in mathematical logic (e.g., $\in$ for "is an element of", $\subset$ for "is a subset of", $\exists$ "there exists", $\forall$ "for all")
- His notation improved the clarity and conciseness of logical expressions
- Founded the journal *Rivista di Matematica* and compiled the *Formulario Mathematico*, an ambitious project to express all known mathematics in a symbolic language

The architect of formal systems: Peano demonstrated how foundational mathematical structures could be rigorously defined by a few axioms, and his notation became indispensable for formalising mathematics

Russell & Whitehead

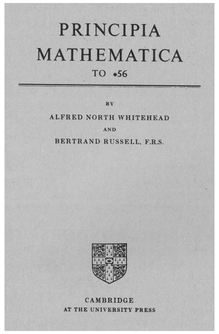
Bertrand Russell (1872–1970) & **Alfred North Whitehead** (1861–1947):

- **Russell**: British philosopher, logician, mathematician, and social critic. His discovery of Russell's paradox directly challenged Frege's work
- **Whitehead**: British mathematician and philosopher. Initially Russell's teacher, then collaborator
- Shared the **logicist belief**: all of mathematics, particularly arithmetic, could be derived from fundamental logical principles and definitions, with set theory as an intermediary

Russell & Whitehead

The monumental *Principia Mathematica* (1910–1913):

- A three-volume work, over 2,000 pages, representing the zenith of the logicist program
- Its goal was to provide a **unified, formal, and axiomatic foundation for all of mathematics** based purely on logical primitives
- Introduced a vast array of new logical symbols and a precise, step-by-step derivation of mathematical concepts
- Famously, it took until page 362 of volume I to prove that $1+1=2$



Russell & Whitehead

Addressing the paradoxes: **Theory of types**

- To avoid Russell's Paradox and similar contradictions, they introduced the **theory of types**
- This theory restricts how sets can be formed and how properties can be applied, preventing "self-referential" definitions (e.g., a set cannot contain itself, and properties can only be applied to objects of a lower "type")
- While successful in preventing known paradoxes, it made the system very complex and somewhat counter-intuitive

The pinnacle and pitfalls of logicism: *Principia Mathematica* was an extraordinary intellectual feat, demonstrating the possibility of formalising mathematics. However, its immense complexity and reliance on non-obvious axioms (like the Axiom of Reducibility) highlighted the challenges of the logicist ambition, paving the way for further foundational inquiries

Gödel's incompleteness theorems

Kurt Gödel (1906–1978)
The unforeseen answer to Hilbert's dream:

- Austrian-American, arguably the most important logician since Aristotle
- His work profoundly impacted mathematical logic, philosophy of mathematics, and theoretical computer science
- While chronologically slightly after the "turn of the century" period (published 1931), his theorems directly addressed and fundamentally altered the goals of Hilbert's program

Gödel's incompleteness theorems

Gödel's first incompleteness theorem:

- For any consistent axiomatic system capable of encoding basic arithmetic, there will always be **true statements within that system that cannot be proven within that system**
- In simpler terms: No single consistent axiomatic system can capture all mathematical truths about numbers
- This directly struck at Hilbert's goal of **completeness**

Gödel's second incompleteness theorem:

- A consistent axiomatic system cannot prove its own consistency.
- This delivered another blow to Hilbert's program, particularly his goal of proving the consistency of arithmetic using finitary methods **within** the system itself

Gödel's incompleteness theorems

A profound paradigm shift: Gödel's theorems demonstrated inherent limitations to formal axiomatic systems, showing that mathematics, despite its rigour, contains irreducible aspects of truth that cannot be fully captured by any single, consistent formal framework. This led to a re-evaluation of the nature of mathematical truth and proof

Henri Lebesgue

The limitations of Riemann integration:

- The Riemann integral (developed by Bernhard Riemann in the mid-19th century) works well for continuous or "nicely behaved" functions
- However, with the rise of increasingly complex and "pathological" functions (e.g., Dirichlet function, continuous nowhere differentiable functions from Weierstrass), its limitations became apparent

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

- Key problem: For some functions, the Riemann integral might not exist, or the order of integration and limits could not be interchanged reliably
- It summed areas of vertical "rectangles" under the curve

Henri Lebesgue

Henri Lebesgue (1875–1941)

Lebesgue's breakthrough:
Measuring sets (1902):

- French mathematician, a student of Émile Borel (who pioneered measure theory in more general contexts)
- Instead of slicing the domain (x-axis), Lebesgue conceived of slicing the **range (y-axis)** of the function
- He introduced the concept of **measure**: a systematic way to assign a *size* (length, area, volume) to subsets of space, including very complex ones
- The **Lebesgue integral** defines the integral of a function by partitioning its **range** into tiny intervals and summing the "measure" of the sets in the domain that map into those intervals

Henri Lebesgue

The power of Lebesgue integration:

- Allows for the integration of a far broader class of functions than the Riemann integral
- Provides a robust framework for interchanging limits and integrals (e.g., Lebesgue's dominated convergence theorem)
- Absolutely foundational for:
  - **Functional analysis** (study of function spaces)
  - **Probability theory** (defining continuous probability distributions)
  - **Harmonic analysis** and **partial differential equations**

A new foundation for analysis: Lebesgue's measure theory provided a powerful, flexible, and rigorous generalisation of integration, becoming the standard for 20th century analysis and beyond

Henri Poincaré

The "Last Universalist":

- French mathematician, theoretical physicist, engineer, and philosopher
- Often described as "the last universalist" for his contributions across nearly all branches of mathematics and physics of his time
- His work on celestial mechanics (the three-body problem) led him to entirely new mathematical concepts

Henri Poincaré (1854–1912)

Henri Poincaré

The birth of modern topology:

- Topology is the study of properties of spaces that are preserved under continuous deformations (stretching, bending, but not tearing or gluing)
- Poincaré's seminal work, ***Analysis Situs*** (1895–1904), is considered the foundational text of algebraic topology
- He introduced key concepts:
  - **Homology** and **homotopy**: Tools to distinguish spaces by their "holes" or "connectedness by paths"
  - The **fundamental group**: A way to capture the "loops" in a space
- Unlike geometry (which studies rigidity) or analysis (which studies change), topology studies qualitative properties of shape

Henri Poincaré

Other major contributions:

- **Dynamical systems theory**: His work on the three-body problem pioneered chaos theory and qualitative analysis of differential equations
- **Special relativity**: Independent, near-simultaneous insights into relativity theory (Lorentz transformations)
- **Philosophy of science**: Argued for the conventional nature of geometry and the importance of intuition
- **Poincaré conjecture**: One of the most famous problems in topology (solved in 2003 by Perelman)

A new lens for space:
Poincaré's invention of topology provided mathematics with a powerful new lens to understand shape and connectedness, independent of size or rigidity, opening up vast new areas of research

A foundation re-examined, a future unveiled

From crises to new realms of inquiry

- **Foundational reckoning**: The seemingly robust mathematics of the 19^{th} century faced its first major crisis with the discovery of **paradoxes in set theory** (e.g., Russell's paradox), prompting a deep re-evaluation of its logical underpinnings
- **The axiomatic imperative**: **Hilbert's program** provided an ambitious vision for a consistent and complete mathematics, driving the development of **mathematical logic** (Frege, Peano, Russell & Whitehead's *Principia Mathematica*)
- **Limits and new paradigms: Gödel's incompleteness theorems**, while coming slightly later, revealed inherent limitations to formal systems, profoundly shifting the understanding of mathematical truth
- **Seeds of new disciplines**: Simultaneously, groundbreaking work by **Lebesgue** (measure theory) and **Poincaré** (topology) established entirely new branches of mathematics that would flourish in the coming century

A foundation re-examined, a future unveiled

The turn of the century was a period of intense intellectual ferment. Mathematics emerged from this foundational turmoil not broken, but profoundly transformed, armed with new tools and a deeper understanding of its own nature. It was now ready to embark on its most explosive century of growth, expanding into an incredibly diverse and interconnected landscape of theories and applications

References

The main sources for this lecture are

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)

This incredible period has been the source of inspiration for many popular books. Among those, we signal the highly enjoyable *Douglas R. Hofstadter*, Godel, Escher, Bach: An Eternal Golden Braid, Basic Books (1999)

History of Mathematics: Lecture 14

Syllabus:

The 20th century in Mathematics:

- Consolidation, abstraction, and new structures

Consolidation and abstraction

From foundations to vast new landscapes
The dawn of a new era:

- The 20th century witnessed an unparalleled expansion and diversification of mathematical knowledge
- Building on the rigour and abstraction of the 19th century, and the foundational debates of the turn of the century
- Mathematics becomes increasingly abstract, interconnected, and essential to other sciences

Consolidation and abstraction

Key focus for this lesson (Early 20th century):

- **Consolidation and axiomatisation**:
  Establishing secure foundations (ZFC, Bourbaki)
- **Rise of functional analysis**:
  Extending calculus to infinite-dimensional spaces
- **Algebraic geometry takes off**:
  From classical curves to abstract varieties

The turn of the century's legacy

- **Foundational crisis resolved**? Russell's paradox exposed flaws in naive set theory. The quest for solid axioms began
- **Hilbert's grand program**: A vision for a consistent, complete, and decidable mathematics, driving much research
- **Mathematical logic takes shape**: Frege, Peano, Russell & Whitehead provided the formal languages and axiomatic methods
- **Gödel's profound limits**: Incompleteness theorems showed that Hilbert's full dream was unattainable, leading to a deeper understanding of mathematical truth
- **New fields emerging**: Measure theory (Lebesgue) and topology (Poincaré) pointed to novel ways of understanding space and quantity

The early 20th century inherited a paradox-stricken but highly formalised mathematical landscape. The task now was to use these new logical tools to rebuild foundations rigorously, while simultaneously exploring the vast new abstract spaces that the 19th century had only just glimpsed

Rebuilding foundations: Axiomatic set theory

The need for restricted set formation:

- Russell's paradox (and others) demonstrated that the "naive" idea that any definable collection forms a set leads to contradictions
- The problem lay in allowing "too large" or "self-referential" sets
- The solution: Restrict the ways in which sets can be formed using a carefully chosen set of axioms

Ernst Zermelo (1871–1953) and **Abraham Fraenkel** (1891–1965):

- **Zermelo** (1908) proposed the first rigorous axiomatic system for set theory to avoid the known paradoxes
- **Fraenkel** (1922) and Thoralf Skolem independently proposed refinements, leading to what is now known as **Zermelo-Fraenkel (ZF) set theory**
- The inclusion of the **axiom of choice** gives us **ZFC**, the standard axiomatic system for mathematics today

Rebuilding foundations: Axiomatic set theory

Key principles of ZFC:

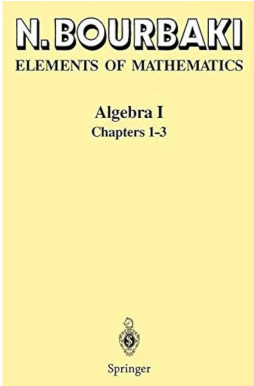
- **Axiom of separation**: Allows forming subsets from existing sets using a property. This prevents Russell's paradox by disallowing the "set of all sets not containing themselves"
- **Axiom of union/power set**: Allows building larger sets from smaller ones
- **Axiom of infinity**: Guarantees the existence of at least one infinite set (e.g., the natural numbers)
- **Axiom of choice**: States that for any collection of non-empty sets, it is possible to choose exactly one element from each set, even if there's no rule for choosing. (Controversial for its non-constructive nature)

The default foundation: ZFC provides a powerful, widely accepted, and (as far as we know) consistent framework for almost all of modern mathematics, effectively resolving the paradoxes that plagued naive set theory

The structuralist program

A fictional name, a real revolution:

- **Nicolas Bourbaki** is the collective pseudonym of a group of mostly French mathematicians, formed in 1935
- Included influential figures like André Weil, Claude Chevalley, Jean Dieudonné, and Henri Cartan
- Their original aim was to write a rigorous, comprehensive treatise presenting the whole of modern mathematics from an axiomatic, foundational point of view



A volume from Bourbaki's *Éléments de mathématique*

The structuralist program

The structuralist approach:

- Mathematics is viewed as the study of **abstract structures**
- These structures are defined axiomatically (e.g., groups, rings, fields, topological spaces, vector spaces)
- Emphasised the relationships and analogies between different mathematical areas through shared underlying structures
- Their influential *Éléments de mathématique* series systematically built up mathematics from set theory, using a highly formal and abstract style

The structuralist program

Impact and legacy:

- Immense influence on university mathematics curricula globally, emphasising abstract algebra, topology, and functional analysis
- Popularised the axiomatic method and the concept of "structure" as central to mathematics
- Standardised notation and terminology (e.g., $\emptyset$ for empty set, $\mathbb{R}$ for real numbers, terms like "injective", "surjective", "bijective")
- While sometimes criticised for excessive abstraction and lack of intuition, their project fundamentally shaped the landscape of pure mathematics

The architects of modern mathematics' structure: Bourbaki's systematic, axiomatic, and structuralist approach provided a unified framework and a common language for diverse mathematical fields, defining how much of 20th century mathematics would be organised and taught

General topology

Beyond metric spaces:

- The 19th century saw the development of **metric spaces** (Fréchet, Hausdorff, Hilbert), where a distance function (metric) defines "closeness"
- However, not all notions of "closeness" or "convergence" require a specific distance
- The goal was to generalise the concepts of continuity, limits, and convergence to the most abstract possible setting

General topology

Felix Hausdorff

Maurice Fréchet (1878–1973)
Felix Hausdorff (1868–1942):

- **Fréchet** (French) in his 1906 thesis defined abstract metric spaces and also initiated the study of general topological spaces, though his definition was not yet the modern one
- **Hausdorff** (German) in his 1914 classic textbook *Grundzüge der Mengenlehre* (Foundations of Set Theory) gave the first formal definition of a **topological space** using axioms for "neighbourhoods" or "open sets"
- This established topology as an independent mathematical discipline

General topology

The axioms of a topological space:

- A set X with a collection of subsets $\mathscr{T}$ (called "open sets") satisfying:
  1. The empty set $\emptyset$ and X itself are in $\mathscr{T}$
  2. Any (finite or infinite) union of sets in $\mathscr{T}$ is in $\mathscr{T}$
  3. Any **finite** intersection of sets in $\mathscr{T}$ is in $\mathscr{T}$
- This simple set of axioms allows for a highly general study of continuity, compactness, and connectedness
- Many different "topologies" can be defined on the same set.

The ultimate abstraction of proximity: General topology provided the most abstract and flexible framework for studying notions of closeness and continuity, becoming fundamental for functional analysis, geometry, and many other areas of 20th century mathematics

Functional analysis

Extending concepts from finite dimensions:

- Recall linear algebra: the study of vector spaces (like $\mathbb{R}^n$) and linear transformations (matrices)
- Functional analysis extends these ideas to spaces where "vectors" are functions, and these spaces can have **infinite dimensions**
- Key idea: Functions can be viewed as points in an abstract vector space

Functional analysis

Origins and motivations:

- Arose from the study of integral equations (e.g., Volterra, Fredholm) and problems in mathematical physics (e.g., calculus of variations)
- The need to understand infinite series, Fourier analysis, and solutions to differential equations more deeply
- The development of abstract measure theory and general topology provided the necessary rigorous framework

Central objects of study:

- **Function spaces**: Collections of functions with specific properties (e.g., continuous functions, square-integrable functions)
- **Linear operators**: Transformations between function spaces (generalising matrices on usual vector spaces)
- Concepts like continuity, convergence, and completeness generalised to these abstract spaces

Functional analysis

Why infinite dimensions? Example: Fourier series

- A function $f(x)$ can be represented as an infinite sum of sines and cosines:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

- Each $\cos(nx)$ and $\sin(nx)$ can be seen as a "basis vector" in an infinite-dimensional space
- The coefficients a_n, b_n are the "coordinates" along these basis vectors
- This representation is only possible in an infinite-dimensional setting

Bridging analysis and algebra: Functional analysis provides powerful tools to solve problems in diverse areas, from quantum mechanics to signal processing, by abstracting analytical problems into the language of linear algebra in infinite dimensions

Hilbert spaces

Generalising Euclidean space:

- In finite-dimensional Euclidean space ($\mathbb{R}^n$), we have notions of:
  - Vectors and vector addition
  - Scalar multiplication
  - Dot product (inner product) for length and angle
  - Distance and completeness
- A **Hilbert space** is an abstract vector space that extends all these geometric notions to infinite dimensions
- It's a complete inner product space

Hilbert spaces

Key features:

- **Inner product**: Allows defining length (norm) and angle between vectors In $\mathbb{R}^n$, the usual inner product is defined as $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, so $||v|| = \sqrt{\langle v, v \rangle}$. In general, $\langle \_, \_ \rangle : V \times V \to \mathbb{R}$ is a function such that $\langle x, x \rangle > 0$ when $x \neq 0$, $\langle x, x \rangle = 0$ when $x = 0$, $\langle x, y \rangle = \langle y, x \rangle$, and $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$. On $\mathbb{C}$, the same holds but $\langle x, y \rangle = \overline{\langle y, x \rangle}$ The angle between x and y is defined as

$$\arccos\left(\frac{\langle x, y \rangle}{||x|| \cdot ||y||}\right)$$

- **Completeness**: Guarantees that sequences that "should" converge, do converge within the space. This is crucial for calculus-like operations
- **Separability**: Often means there's a countable "basis" of functions

Hilbert spaces

Example: L^2 space

- The space of square-integrable functions (functions f such that $\int |f(x)|^2 dx < \infty$)
- The inner product (on $\mathbb{C}$) is defined as $\langle f, g \rangle = \int f(x)\overline{g(x)}dx$
- This space is a central example of a Hilbert space

Hilbert spaces

Profound importance in physics:

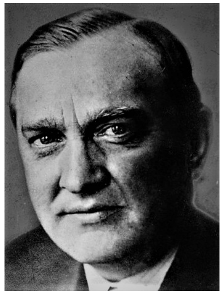
- **Quantum mechanics**: The state space of a physical system in quantum mechanics is represented by a Hilbert space
- Wave functions (describing particles) are elements of a Hilbert space
- Observables (like position, momentum, energy) are represented by linear operators on this space
- The "probability interpretation" relies heavily on the inner product (measuring amplitudes)

The mathematical home of quantum mechanics: Hilbert spaces provided the robust mathematical framework for quantum mechanics, allowing for the rigorous treatment of wave functions, operators, and quantum phenomena

Banach spaces and linear operators

Beyond Hilbert spaces: Banach spaces

- While Hilbert spaces are crucial for their geometric structure (inner product), not all important function spaces possess one
- A **Banach space** is a complete vector space equipped with a norm
  - It has a **norm** (a concept of "length" or "size" for vectors/functions)
  - It is **complete** (all Cauchy sequences converge within the space, preventing "holes")
- Unlike Hilbert spaces, it does *not* necessarily have an inner product or a notion of angle
- **Stefan Banach** (1892–1945), a leading figure of the Lwów School of Mathematics, was central to developing this theory



Stefan Banach

Banach spaces and linear operators

Examples of Banach spaces:

- L^p spaces (for $p \geq 1$): Spaces of functions whose p-th power of their absolute value is integrable.
  (L^2 is a special case that is also a Hilbert space)
- $C[a,b]$: The space of continuous functions on an interval $[a,b]$, with the supremum norm ($||f|| = \max|f(x)|$)

Banach spaces and linear operators

Linear operators: The "matrices" of infinite dimensions

- A **linear operator** is a linear transformation between vector spaces (e.g., from one Banach space to another)
- They generalise matrices from finite to infinite dimensions
- Many fundamental operations in analysis are linear operators:
  - Differentiation: $D(f) = f'$
  - Integration: $I(f) = \int_a^x f(t)dt$
- **Spectral theory**: The study of eigenvalues and eigenvectors of linear operators, generalising diagonalising matrices. This is crucial for understanding the behaviour of operators and has applications in quantum mechanics, differential equations, and signal processing

A universal language for analysis: Banach spaces and the theory of linear operators provided a powerful and abstract framework for tackling a vast range of problems, expanding the scope and generality of analytical methods

Algebraic geometry

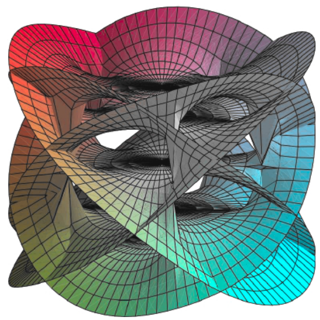
Classical roots: Geometry from polynomials

- Algebraic geometry traditionally studies geometric shapes (curves, surfaces, and their higher-dimensional analogues) that are defined by systems of polynomial equations
- Example: A circle ($x^2 + y^2 = 1$), a parabola ($y = x^2$), or a sphere ($x^2 + y^2 + z^2 = 1$)
- Problems often involved classifying these shapes, understanding their intersections, and identifying their singular points
- This field dates back to ancient Greece (conic sections) and flourished with Descartes' analytical geometry in the 17th century

Algebraic geometry

The 19th century and early abstraction:

- Great progress in the 19th century (e.g., Riemann surfaces in complex analysis, projective geometry)
- Introduction of complex numbers and projective space simplified many classical theorems
- However, the tools were often ad-hoc, relying on intuition or specific coordinate systems
- The desire for a more rigorous, general, and coordinate-free approach grew



An algebraic curve defined by a polynomial equation

Algebraic geometry

The 20th century transformation: Abstraction and algebraisation

- The foundational crises and the success of axiomatic methods (like ZFC) inspired a push for deeper abstraction
- Focus shifted from geometric intuition alone to the underlying algebraic structures (rings, fields, ideals) associated with geometric objects
- The goal became to define "spaces" (called **varieties** and later **schemes**) purely algebraically, then to study their geometric properties through their algebraic structure
- This allowed for the study of "geometry" over any field (finite fields, p-adic numbers), not just real or complex numbers

A symphony of abstraction: Algebraic geometry transformed into a highly abstract and powerful field, creating deep connections between commutative algebra, number theory, topology, and complex analysis, becoming a central pillar of modern pure mathematics

Hilbert's algebraic foundations for geometry

Setting the stage for abstract algebraic geometry:

- While not strictly an "algebraic geometer" in the modern sense, **David Hilbert**'s work in invariant theory and commutative algebra provided crucial foundational theorems
- His axiomatic approach and insistence on rigorous proof profoundly influenced the field

Hilbert's algebraic foundations for geometry

Key theorems with geometric ramifications:
Hilbert's basis theorem (1890):

- States that every ideal in a polynomial ring over a field (or a Noetherian ring) is finitely generated
- Geometrically: This means that any algebraic variety (set of zeros of polynomials) can be defined by a *finite* number of polynomial equations, even if it initially seems to require an infinite number. This was a critical finiteness result

Hilbert's algebraic foundations for geometry

Key theorems with geometric ramifications:
Hilbert's nullstellensatz (Zeros theorem, 1893):

- A fundamental bridge between algebra and geometry
- It relates ideals in polynomial rings to algebraic varieties. For example, it establishes a correspondence between algebraic sets and radical ideals
- Geometrically: If a polynomial vanishes on all the common zeros of a set of other polynomials, then it is "algebraically dependent" on them in a specific way

Let k be a field and K be an algebraically closed field extension of k. Consider the polynomial ring $k[x_1,\dots,x_n]$ and let J be an ideal in this ring. Let $\mathrm{V}(J) = \{a \in K^n \mid f(a) = 0 \text{ for all } f \in J\}$

Hilbert's nullstellensatz states that if p is a polynomial in $k[x_1,\dots,x_n]$ such that $p(a) = 0$ for all $a \in \mathrm{V}(J)$, then there exists $r \in \mathbb{N}$ such that $p^r \in J$

Hilbert's algebraic foundations for geometry

The twentieth century challenge: Geometry over any field

- Classical algebraic geometry often focused on complex numbers ($\mathbb{C}$)
- Questions arose about doing "geometry" over other fields, especially **finite fields** (e.g., $\mathbb{Z}_p$), crucial for number theory and cryptography
- Counting the number of points on algebraic varieties over finite fields became a central, incredibly difficult problem, linking geometry, algebra, and number theory

Algebra as the foundation: Hilbert's theorems provided the essential algebraic machinery that allowed algebraic geometers to move beyond intuitive geometric reasoning and build the field on a solid, abstract algebraic bedrock, enabling generalisations to vastly different contexts

The Weil conjectures

André Weil (1906–1998) and the challenge:

- French mathematician, one of the founders of the Bourbaki group
- In 1949, **André Weil** formulated three profound conjectures (and an analogue of the Riemann hypothesis) concerning the number of solutions to polynomial equations over finite fields
- These conjectures provided a concrete target for algebraic geometers, promising deep insights into number theory
- They connected the counting of points on varieties over finite fields to complex analysis and topology, via an analogue of the Riemann zeta function

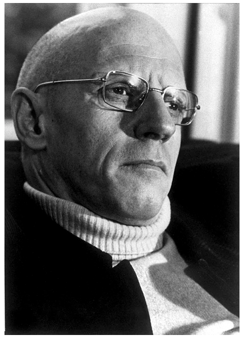
André Weil

The Weil conjectures

A catalyst for new theories:

- Proving these conjectures required the development of entirely new, highly abstract mathematical theories
- The existing "classical" tools of algebraic geometry were insufficient

The Weil conjectures



Alexandre Grothendieck

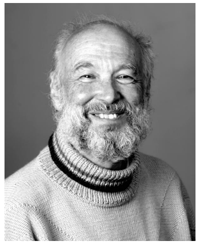
The revolutionary work of Alexandre Grothendieck (1928–2014):

- A visionary French-German mathematician, considered one of the greatest in 20th century
- To prove the Weil conjectures, Grothendieck single-handedly revolutionised algebraic geometry, introducing the theory of **schemes**
- **Schemes**: An extreme generalisation of algebraic varieties, allowing for "geometric" intuition to be applied to a much broader class of objects (e.g., prime ideals in rings), creating bridges to number theory
- He developed **étale cohomology**, a sophisticated cohomology theory necessary to prove the conjectures

The Weil conjectures

Pierre Deligne (1944–): The final proof

- Building on Grothendieck's vast framework, **Pierre Deligne** (1974) successfully proved the last and most difficult of the Weil conjectures (the Riemann hypothesis analogue)



A triumph of abstraction and interconnectedness: The Weil conjectures, and their eventual proofs, exemplify the power of extreme abstraction in 20th century mathematics, demonstrating how deep problems can drive the creation of entirely new theories that ultimately connect disparate fields like number theory, geometry, and topology

Expanding abstraction

From rigour to unforeseen realms

- **Solidifying the base**: The early 20th century successfully provided **rigorous foundations** for mathematics with axiomatic set theory (**ZFC**) and the structuralist approach of **Bourbaki**
- **Generalising space**: The concept of "space" was profoundly abstracted with the development of **General topology** (Hausdorff), providing a universal framework for continuity
- **Infinite dimensions of analysis**: **Functional analysis** emerged as a powerful tool to study functions as "vectors" in infinite-dimensional spaces (**Hilbert** and **Banach** spaces), crucial for physics and PDEs
- **Abstracting geometry**: **Algebraic geometry** underwent a deep transformation, using sophisticated algebraic tools (Hilbert's theorems, Weil conjectures driving **Grothendieck**'s schemes) to study geometric objects over various fields

Expanding abstraction

From rigour to unforeseen realms *The early 20th century was defined by a drive for ultimate rigour and abstraction. Mathematicians successfully rebuilt logical foundations and developed entirely new, sophisticated languages — functional analysis, modern algebraic geometry — to explore structures previously unimaginable. This period of deep theoretical construction laid the groundwork for the explosion of interconnected research fields and grand problem-solving efforts that would characterise the mid-to-late 20th century*

References

The main sources for this lecture are

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)

In addition, some aspects have been taken from *Piergiorgio Odifreddi*, La matematica del Novecento: Dagli insiemi alla complessità, Einaudi (2000). However, in many cases the sources are drawn from the instructor's own expertise

History of Mathematics: Lecture 15

Syllabus:

The 20th century in Mathematics:

- Interconnections and problem solving

Interconnections and problem solving

Mathematics as a web of theories

The mid-to-late 20th century landscape:

- Building on the rigorous foundations and abstract structures from the early 20th century
- A period characterised by a flourishing of deep connections between seemingly disparate fields
- Mathematics becomes a powerful tool for solving long-standing problems and driving scientific progress

Key focus for this lesson:

- **Algebraic topology flourishes**: Using algebra to study geometric shapes
- **Number theory's grand challenges**: Tackling ancient and modern problems with new techniques

Early 20th century foundations

Tools acquired, structures defined

- **Solid axioms**: ZFC set theory and Bourbaki's structuralism provided a unified, rigorous language
- **Generalised spaces**: General topology defined abstract notions of closeness and continuity
- **Infinite-dimensional analysis**: Functional analysis (Hilbert, Banach spaces, operators) opened new analytical avenues
- **Abstract algebra & geometry**: Algebraic geometry's transformation (Hilbert, Weil, Grothendieck) and deep group theory (CFSG, Lie groups) laid powerful algebraic groundwork

The first half of the 20th century provided mathematics with a robust, abstract framework. The mid-to-late century would see these sophisticated tools applied to tackle profound problems, revealing unexpected symmetries and connections across the mathematical landscape

Group theory deepens

Building blocks of symmetry: Finite simple groups

- Recall group theory (from Galois to Klein) as the study of symmetry
- Just as prime numbers are the building blocks for integers, **simple groups** (non-trivial groups whose only normal subgroups are the group itself and the trivial group) are the fundamental, irreducible building blocks for all finite groups (Jordan-Hölder theorem)
- **The classification of finite simple groups (CFSG):**
  - A monumental, collaborative effort spanning over 50 years (roughly 1955–2004) by hundreds of mathematicians
  - Resulting in a proof spanning thousands of journal pages
  - It classifies all finite simple groups into several families (e.g., cyclic groups of prime order, alternating groups, groups of Lie type) and **26 "sporadic" groups** (which don't fit into obvious families, like the "monster group")
- This was one of the largest mathematical projects in history, demonstrating the power of collaborative, systematic classification

Group theory deepens

Continuous symmetries: Lie groups and Lie algebras

- Named after Norwegian mathematician **Sophus Lie** (1842–1899)
- **Lie groups** are groups that are also smooth manifolds, meaning they describe continuous symmetries
- Examples: Rotations in 3D space (SO(3)), Lorentz transformations in special relativity
- The study of Lie groups often involves their associated **Lie algebras**, which are simpler, linear algebraic objects that capture the local structure of the Lie group
- **Profound connections**:
  - **Differential geometry**: Symmetries of manifolds
  - **Physics**: Fundamental to quantum mechanics (e.g., symmetries of particles), particle physics (e.g., standard model's gauge symmetries), and general relativity

Group theory deepens

Symmetry: From building blocks to fundamental forces

The 20th century saw group theory provide an exhaustive catalogue of finite symmetries and a robust framework for understanding continuous symmetries, becoming indispensable for both pure mathematics and theoretical physics.

Algebra to the rescue of geometry

Building on Poincaré's vision:

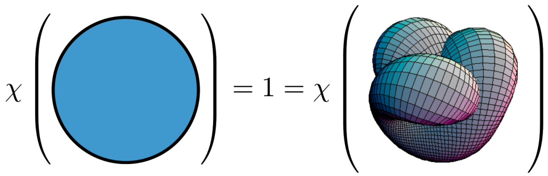
- Recall Poincaré's foundational work in *Analysis Situs* (1895–1904), introducing concepts like homology and the fundamental group
- Algebraic topology is the study of topological spaces using tools from abstract algebra
- The core idea: Associate algebraic invariants (groups, rings, modules) to topological spaces. These invariants don't change if the space is continuously deformed

Algebra to the rescue of geometry

Why algebra for geometry?

- Geometric objects can be incredibly complex (e.g., high-dimensional manifolds)
- Pure intuition often fails beyond 3 dimensions
- Algebraic structures (e.g. groups) are easier to compute with and classify
- If two spaces have different algebraic invariants, they cannot be topologically equivalent

Algebra to the rescue of geometry



Topological space and its algebraic representation

The mid-20th century flourishing:

- Rapid development, particularly after World War II, fuelled by the needs of other fields (differential geometry, number theory, even physics)
- Major figures: Emmy Noether (earlier, but influence on cohomology), Heinz Hopf, Norman Steenrod, Samuel Eilenberg, Saunders Mac Lane

Algebra to the rescue of geometry

Key algebraic invariants (conceptual):

- **Homotopy groups** Generalise the fundamental group (Poincaré) to higher dimensions (π_n). They capture how n-dimensional spheres can be continuously mapped into the space
- **Homology & cohomology theories**:
  - Provide algebraic descriptions of how the "holes" in a space are related in various dimensions
  - Cohomology groups often have a richer algebraic structure (e.g., a ring structure) than homology groups, making them more powerful
- These theories allow mathematicians to classify and understand highly complex shapes like knots, surfaces, and higher-dimensional manifolds

The power of translation: Algebraic topology acts as a powerful dictionary, translating complex geometric problems into manageable algebraic questions, which then yield profound insights into the structure and properties of spaces

Fixed-point theorems

Interconnections across mathematics:

- Algebraic topology is not just about abstract spaces; its tools are vital across many fields:
  - **Differential geometry**: Classifying manifolds, characteristic classes
  - **Algebraic geometry**: The cohomology theories (e.g., étale cohomology mentioned with Grothendieck) used to solve the Weil conjectures have topological roots
  - **Number theory**: Connecting to arithmetic properties
  - **Functional analysis**: Studying properties of operators

Fixed-point theorems

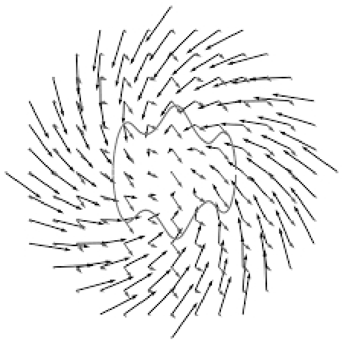
A major class of results: Fixed-point theorems

- A **fixed point** of a function $f(x)$ is a value x such that $f(x) = x$
- Fixed-point theorems provide conditions under which a continuous function must have at least one fixed point
- These theorems are incredibly powerful for proving the existence of solutions to equations in areas where direct computation is impossible

Fixed-point theorems

L.E.J. Brouwer (1881–1966) and the first theorem:

- Dutch mathematician, also a key figure in intuitionism (a school of thought on foundations of math, briefly mentioned earlier)
- **Brouwer fixed-point theorem** (1911): Any continuous function from a closed disk (or solid ball) to itself must have at least one fixed point
- Intuitive example: If you stir a cup of coffee, there's always at least one point in the coffee that ends up in its original position



Brouwer's fixed-point theorem in 2D

Fixed-point theorems

Solomon Lefschetz (1884–1972) and generalisation:

- American mathematician of Russian-Jewish origin
- **Lefschetz fixed-point theorem** (1926): A much more general theorem that uses homology groups to count fixed points (with multiplicity)
- It applies to a broader class of spaces and provides a criterion involving algebraic invariants

Beyond intuition: guaranteed existence: Fixed-point theorems, born from algebraic topology, are not just elegant results; they are fundamental tools guaranteeing the existence of solutions in diverse fields from economics (e.g., existence of market equilibrium) to differential equations

Diophantine equations

Diophantine equations: Ancient puzzles, enduring mysteries

- Polynomial equations where we seek only **integer solutions**
- Named after Diophantus of Alexandria (3^{rd} century CE)
- Examples:
  - Pythagorean triples: $x^2 + y^2 = z^2$ (e.g., $3^2 + 4^2 = 5^2$)
  - Fermat's Last Theorem: $x^n + y^n = z^n$ for $n > 2$ (no non-trivial integer solutions)
- Finding solutions or proving their non-existence can be incredibly difficult

Diophantus of Alexandria

Diophantine equations

Hilbert's tenth problem (1900): The decidability question

- One of David Hilbert's famous 23 problems posed at the 1900 International Congress of Mathematicians
- It asked for an **algorithm** that could determine, in a finite number of steps, whether a given Diophantine equation has integer solutions
- Hilbert believed such an algorithm must exist (a testament to his formalist optimism)

Diophantine equations

The surprising resolution: Yuri Matiyasevich (1970)

- Decades of work by Julia Robinson, Martin Davis, and Hilary Putnam laid the groundwork
- In 1970, Soviet mathematician **Yuri Matiyasevich** proved that **no such general algorithm exists**
- This means Hilbert's tenth problem is **undecidable**
- **Connection to computability theory**: Matiyasevich's proof relied fundamentally on concepts from computability theory (e.g., Turing machines, recursively enumerable sets), a field that developed much later than Hilbert posed his problem

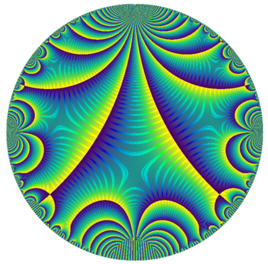
Limits of computability: A profound interconnection

The resolution of Hilbert's tenth problem demonstrated an inherent limitation to what algorithms can achieve and exemplified the unexpected connections between seemingly disparate fields — number theory and the nascent field of theoretical computer science

Modular forms

What are modular forms?
(A conceptual view):

- They are complex analytic functions that are defined on the upper half-plane of complex numbers
- Their defining characteristic is an incredible amount of **symmetry** under a specific group of transformations, forming the modular group
- They have a specific growth condition and a 'nice' behaviour at "infinity"



Symmetry in the domain of the modular group

- Despite their abstract definition, their Fourier coefficients (the numbers appearing in their series expansion) encode profound arithmetic information about integers

Modular forms

Mysterious connections emerge:

- Discovered in the 19^{th} century by Dedekind, Riemann, Poincaré and Klein, their deeper significance became apparent in the 20^{th} century
- They initially seemed to be fascinating but isolated objects
- However, mathematicians noticed mysterious and surprising links between their coefficients and various number-theoretic quantities (e.g., partition functions, number of ways to write an integer as a sum of squares)
- One of the most unexpected links was to **elliptic curves**

Modular forms

Elliptic curves: Another bridge

- An **elliptic curve** is a curve defined by an equation of the form $y^2 = x^3 + ax + b$ (with certain conditions)
- They are fundamental objects in number theory and cryptography
- The number of points on an elliptic curve over finite fields (recall our discussion of Weil conjectures) holds deep arithmetic information

The Taniyama-Shimura-Weil conjecture (later theorem):

- This incredibly profound conjecture, dating back to the 1950s, proposed that **every elliptic curve over the rational numbers is associated with a unique modular form**
- It was a speculative idea, suggesting a deep, hidden connection between two seemingly unrelated branches of mathematics

Modular forms

From abstract symmetry to arithmetic truths: Modular forms, initially studied for their rich symmetries, emerged as a Rosetta Stone, translating deep arithmetic properties of numbers and elliptic curves into the language of complex analysis, setting the stage for a grand resolution

Fermat's last theorem

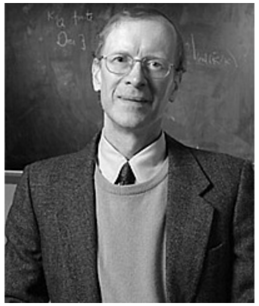
The problem: Simple to state, impossible to prove (for centuries)

- Recall **Fermat's last theorem** (FLT): No three positive integers a, b, c can satisfy $a^n + b^n = c^n$ for any integer value of n greater than 2
- Posed by Pierre de Fermat in 1637, famously claimed he had a "truly marvellous proof" but no room to write it
- For 350 years, it stood as a tantalising challenge, resisting all attempts by the greatest mathematicians

Fermat's last theorem

The crucial link: From elliptic curves to modular forms

- In the mid-1980s, Ken Ribet and Gerhard Frey linked FLT to the **Taniyama-Shimura-Weil (TSW) conjecture**
- Frey suggested that if a counterexample to FLT existed, it would lead to an elliptic curve that could *not* be associated with a modular form, thus contradicting the TSW conjecture
- Ribet proved Frey's "epsilon conjecture" in 1986: if TSW is true, then FLT must be true
- Suddenly, solving a 350-year-old number theory problem became equivalent to proving a major conjecture in algebraic geometry and modular forms



Sir Andrew Wiles

Fermat's last theorem

Andrew Wiles: The monumental proof

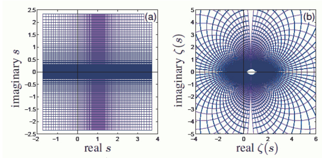
- **Sir Andrew Wiles** (1953–), a British mathematician at Princeton University, embarked on a secret seven-year quest to prove the TSW conjecture for a large class of elliptic curves
- He leveraged decades of deep mathematical development in elliptic curves, modular forms, Galois representations, and Iwasawa theory
- His initial announcement in 1993 was followed by a small gap, which he fixed with Richard Taylor, publishing the final proof in 1995
- **Significance**: Not only did it prove FLT, but it validated a vast amount of abstract mathematics and demonstrated profound connections between seemingly disparate fields

A triumph of interconnectedness: The proof of Fermat's Last Theorem stands as one of the great intellectual achievements of the 20th century, showcasing how deep problems require crossing traditional boundaries and leveraging the interconnected strength of modern mathematical theories

The Riemann hypothesis

The prime numbers and zeta function

- From our 19th century discussion: Bernhard Riemann's groundbreaking 1859 paper on the distribution of prime numbers
- He connected the primes to the behaviour of the **Riemann zeta function** $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$
- This function, initially defined for $\mathrm{Re}(s) > 1$, can be analytically continued to the entire complex plane

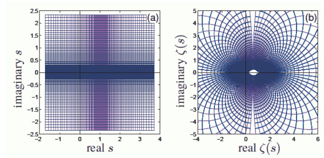


The critical line for the Riemann zeta function

The Riemann hypothesis

The hypothesis:

- The **Riemann hypothesis (RH)** states that all non-trivial zeros of the Riemann zeta function lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$
- (The "trivial" zeros are at negative even integers: $-2, -4, -6, \ldots$)



The critical line for the Riemann zeta function

The Riemann hypothesis

Why is it so important?

- Its truth would imply profound results about the **distribution of prime numbers**. For example, it would give very tight bounds on the error term in the prime number theorem
- It has deep connections to many other areas of mathematics, including analysis, number theory, and even physics (e.g., random matrix theory)
- Thousands of theorems have been proven **assuming** the Riemann hypothesis is true. If it were proven false, a significant portion of mathematics would need re-evaluation

The Riemann hypothesis

A Millennium prize problem:

- The Clay Mathematics Institute listed the Riemann hypothesis as one of its seven Millennium Prize problems in 2000, with a $1 million prize for its solution
- It remains one of the most challenging and important unsolved problems in mathematics

The unsolved heart of number theory: The Riemann hypothesis continues to drive research in analytic number theory and beyond, serving as a powerful reminder that fundamental mysteries still lie at the core of mathematics, inspiring new tools and connections in the quest for truth

Geometry as gravity

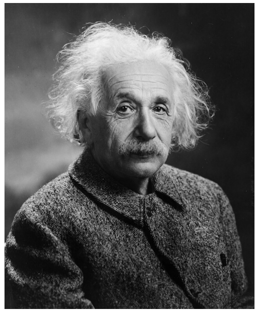
Recap: Geometry beyond flatness (19^{th} century)

- Recall Gauss's intrinsic geometry of surfaces and Riemann's generalisation to higher dimensions
- Key idea: Curvature can be an intrinsic property of a space, not just how it bends in higher dimensions
- Mathematicians developed tools to study differentiable manifolds and their metric properties (distances, angles, volumes) using calculus

Geometry as gravity

Albert Einstein (1879–1955) and General Relativity (1915):

- Einstein's earlier Special Relativity (1905) unified space and time into spacetime
- **General Relativity (GR)** aimed to incorporate gravity into this framework
- Revolutionary insight: Gravity is not a force acting across space, but a manifestation of the **curvature of spacetime** itself
- Mass and energy "tell" spacetime how to curve, and spacetime "tells" mass and energy how to move



Albert Einstein

Geometry as gravity

The indispensable role of Riemannian geometry:

- Einstein realised that the existing tools of physics (Newtonian mechanics, Maxwell's electromagnetism) were insufficient to describe this geometric view of gravity
- He found his needed mathematical language in the advanced differential geometry developed by Riemann, Ricci-Curbastro, and Levi-Civita
- The **Riemann curvature tensor** became central to GR, quantifying how spacetime is curved
- The **Einstein Field Equations** (EFE), which relate spacetime curvature to the distribution of energy and momentum, are expressed entirely in the language of differential geometry

Geometry as gravity

A perfect symbiosis: General Relativity represents a pinnacle of interdisciplinary achievement, demonstrating how abstract mathematical concepts, developed for their own sake, can provide the exact framework needed to describe the fundamental laws of the universe, fundamentally reshaping our understanding of reality

Mathematics as a unified web

From abstraction to interconnected solutions

- **Bridging geometry and algebra**: **Algebraic topology** blossomed, using algebraic invariants (homotopy, homology, cohomology) to classify and understand geometric spaces, leading to powerful existence proofs like **fixed-point theorems**
- **Solving ancient puzzles**: **Number theory's grand challenges** saw the resolution of **Hilbert's 10th problem** (Matiyasevich's undecidability proof linking to computability) and the epic triumph of **Fermat's last theorem** (Wiles, through elliptic curves and modular forms). The **Riemann hypothesis** stands as an ongoing beacon
- **Geometry explaining reality**: **Differential geometry** provided the essential language for physics, e.g., **general relativity**

Mathematics as a unified web

From abstraction to interconnected solutions

- **The pinnacle of abstraction**: Pushing the boundaries of generalisation even further, concepts like Alexandre Grothendieck's **toposes** emerged, offering frameworks that unify logic, set theory, and geometry in unprecedented ways

The mid-to-late 20th century witnessed the extraordinary power of abstract mathematical tools to solve long-standing problems and uncover profound interconnections across disciplines. This period cemented mathematics as a deeply unified and indispensable science, setting the stage for the computational revolution and the exploration of entirely new frontiers that would define the turn of the millennium

References

The sources for this lecture are

- *Carl B. Boyer*, A History of Mathematics, John Wiley & Sons (1968)
- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Piergiorgio Odifreddi*, La matematica del Novecento: Dagli insiemi alla complessità, Einaudi (2000)

However, the most recent aspects come from the direct knowledge of the instructor

History of Mathematics: Lecture 16

Syllabus:

The 20^{th} century in Mathematics:

- Computation, complexity, and new frontiers

Computation, complexity, and new frontiers

Mathematics in the digital age

The late 20^{th} and early 21^{st} century landscape:

- Building upon the vast theoretical frameworks established earlier
- Marked by the rise of computation as both a tool and a field of study in its own right
- Emergence of new areas driven by technology, interdisciplinary needs, and the pursuit of complexity
- Continued focus on grand challenges, both solved and open

Key focus for this lesson:

- **The age of computation**: Theoretical foundations and limits
- **Chaos theory and fractals**: Exploring complexity and non-linearity
- **Major unsolved problems**: A look at the persistent frontiers

A century of abstraction

- **Early 20^{th} century**:
  Focused on **consolidation and abstraction**
  - Rigorous foundations (ZFC, Bourbaki)
  - Generalised spaces (topology, functional analysis)
  - Abstract algebraic structures (modern algebraic geometry, group theory)
- **Mid-to-late 20^{th} century**:
  Characterised by **interconnections and problem-solving**
  - Algebraic topology bridging algebra and geometry
  - Solution of long-standing problems (Fermat's last theorem, Hilbert's 10^{th})
  - Interplay between differential geometry and physics (general relativity)
  - Pinnacle of abstraction (e.g., toposes)

Having built powerful abstract tools and demonstrated their profound interconnectedness, mathematics in the late 20^{th} century was poised for new revolutions, particularly driven by the emergence of computing and the exploration of complex, non-linear phenomena

Alan Turing

The crisis of foundational mathematics (revisited):

- Recall Hilbert's program (1920s): to formalise all of mathematics and prove its consistency and decidability
- Kurt Gödel's incompleteness theorems (1931) showed fundamental limits to formal systems, dashing Hilbert's full dream
- This led to the pressing question: What exactly *is* a computable function or an algorithm? Is there a precise mathematical definition?

Alan Turing

Alan Turing (1912–1954):
A visionary mathematician

- British mathematician, logician, and computer scientist
- Instrumental in formalising the concept of an algorithm and computation
- His work had a profound impact during WWII (code-breaking at Bletchley Park, underlying the "Enigma" machine)

Alan Turing

Alan Turing

The Turing machine (1936): A conceptual breakthrough

- Turing introduced a simple, abstract mathematical model of computation: the **Turing machine**
- It consists of:
  - An infinitely long tape divided into cells
  - A read/write head that moves along the tape
  - A finite set of internal states
  - A set of rules that dictate behaviour based on the current state and symbol under the head
- **Significance**: Despite its simplicity, it can simulate *any* algorithm or computation that a modern computer can perform. It defines precisely what "computable" means

Defining the undefinable: What is computation? Turing's genius lay in providing a rigorous mathematical definition for the intuitive notion of an algorithm, paving the way for the entire field of theoretical computer science and distinguishing between what is computable and what is not

The Church-Turing thesis and undecidability

The Church-Turing thesis (1936): A foundational assertion

- Independently of Turing, American mathematician **Alonzo Church** (1903–1995) developed a different formal system for computation called **lambda calculus**
- Remarkably, Church and Turing (along with others like Stephen Kleene and Emil Post) showed that their different formalisms were **equivalent** in computational power
- **The thesis**: Any function that can be computed by an algorithm (in the intuitive sense) can be computed by a Turing machine (or equivalently, by lambda calculus)
- **Significance**: It's a fundamental belief, not a theorem (as "intuitive sense" isn't formally defined), but it provides the bedrock for theoretical computer science. It posits that Turing machines define the absolute limits of computation

The Church-Turing thesis and undecidability

The uncomputable: Undecidable problems

- A profound consequence of precisely defining "computable" is the realisation that some problems are **uncomputable** or **undecidable**
- This means no algorithm, no Turing machine, can ever solve them for all possible inputs in a finite amount of time

The halting problem:

- Posed by Turing himself: Given an arbitrary program and an input, can we determine if the program will eventually stop (halt) or run forever?
- Turing proved in 1936 that the halting problem is **undecidable**. There is no general algorithm that can answer this question for all possible programs and inputs

The Church-Turing thesis and undecidability

Connecting back: Hilbert's 10th problem:

- As we discussed before, Hilbert's 10th problem asked for an algorithm to determine if a Diophantine equation has integer solutions
- Matiyasevich's theorem (1970) proved this problem is also **undecidable**, relying directly on the work in computability theory initiated by Turing and Church

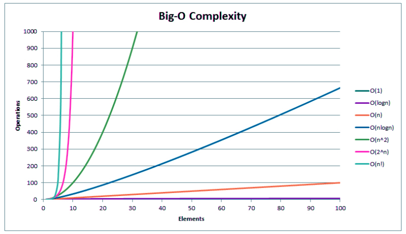
Profound limits: Not all questions have algorithmic answers
The Church-Turing thesis and the discovery of undecidable problems fundamentally changed our understanding of what can be achieved through computation, revealing inherent limitations to algorithmic solutions, even for clearly defined mathematical questions

Computational complexity

Beyond computable: The question of efficiency

- While computability theory (Turing, Church) tells us **if** a problem can be solved by an algorithm, it doesn't say **how long** or **how much memory** it will take
- **Computational complexity theory** studies the resources (time and space) required to solve computational problems
- It's about classifying problems based on their inherent difficulty

Computational complexity



Polynomial vs. exponential time growth

Measuring complexity: Growth rates

- We measure complexity as a function of the input size (n)
- **Polynomial time** (efficient): Algorithms whose running time grows as a polynomial function of n (e.g., n^2, n^3)
- **Exponential time** (intractable): Algorithms whose running time grows exponentially (e.g., 2^n, $n!$)

Computational complexity

Key complexity classes and P vs NP:

- **Class P** (Polynomial time): Contains decision problems for which a solution can be **found** (computed) by a deterministic algorithm in polynomial time. (e.g., sorting a list, multiplying numbers)
- **Class NP** (Non-deterministic Polynomial): Contains decision problems for which a proposed solution can be **verified** by a deterministic algorithm in polynomial time. (e.g., given a list of cities and connections, is there a route that visits all cities exactly once with total length less than X?)
- Every problem in **P** is also in **NP** (if you can find a solution efficiently, you can certainly verify it efficiently)

Computational complexity

The P vs NP problem:

- **Is P = NP**? This is one of the greatest open problems in mathematics and computer science
- It asks: If a solution to a problem can be **quickly verified**, can it also be **quickly found**?
- Most computer scientists believe **P** $\neq$ **NP**

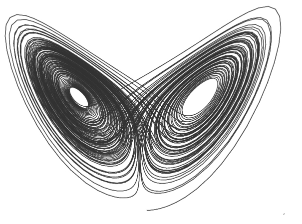
A million-dollar question with profound implications:
The **P** vs **NP** problem has immense practical implications for cryptography, artificial intelligence, optimisation, and nearly every area of science and engineering. Its resolution would redefine the limits of what we consider computationally feasible

Chaos theory: Order in apparent randomness

Beyond predictability: When simple rules yield complex behaviour

- For centuries, scientists believed that if one knew the initial conditions precisely, the future state of a system could be perfectly predicted (Laplace's demon)
- **Chaos theory** emerged to study dynamic systems that, despite being deterministic (governed by fixed rules), exhibit highly unpredictable and seemingly random behaviour
- This unpredictability arises from an extreme sensitivity to initial conditions

Chaos theory: Order in apparent randomness



The Lorenz attractor

"The butterfly effect": Sensitive dependence on initial conditions

- Coined by meteorologist **Edward Lorenz** (1917–2008) in the 1960s
- He discovered that tiny, imperceptible differences in initial input to his weather model led to vastly different long-term predictions

- Metaphor: A butterfly flapping its wings in Brazil could theoretically set off a tornado in Texas weeks later
- This fundamental property means long-term prediction in chaotic systems is often impossible, even with the aid of powerful computers

Chaos theory: Order in apparent randomness

Mathematical tools for chaos:

- Chaos theory utilises tools from:
  - **Dynamical systems**: The study of how systems change over time
  - **Topology**: Understanding the qualitative behaviour of trajectories
  - **Measure theory**: Quantifying probabilities and averages in chaotic systems
- Chaotic systems often settle into strange attractors – complex, fractal-like geometric shapes that describe their long-term behaviour.
  The Lorenz attractor is a famous example

$$\frac{dx}{dt} = 10(y-x)$$
$$\frac{dy}{dt} = x(28-z)-y$$
$$\frac{dz}{dt} = xy - \frac{8}{3}z$$

Chaos theory: Order in apparent randomness

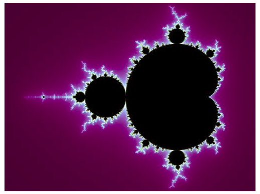
From determinism to unpredictability: Chaos theory revealed that determinism does not always imply predictability, fundamentally altering our understanding of phenomena ranging from weather patterns and fluid dynamics to population growth and even brain activity

Fractals: The geometry of roughness

Beyond Euclidean geometry: A new language for Nature

- Traditional Euclidean geometry describes smooth shapes (lines, circles, spheres, but also parabolas, iperbolas, . . .)
- But what about coastlines, clouds, trees, or mountains? These are rough, irregular, and defy simple description
- Mathematician **Benoit Mandelbrot** (1924–2010) coined the term "**fractal**" in 1975 to describe these shapes

Fractals: The geometry of roughness

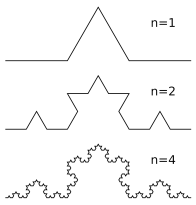


The Mandelbrot set

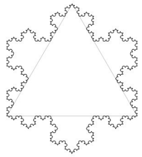
Defining properties of fractals:

- **Self-similarity** A fractal displays similar patterns at increasingly small scales: Zooming in reveals smaller copies of the whole
  A compact topological space X is *self-similar* if there exists a finite set I indexing a family of non-surjective homeomorphisms $\{f_i\}_{i\in I}$ such that $X = \bigcup_{i\in I} f_i(X)$
- **Fractional (or *fractal*) dimension**: Unlike conventional shapes (lines=1D, squares=2D, cubes=3D), fractals often have non-integer dimensions. For example, a coastline might have a fractal dimension between 1 and 2, indicating how "space-filling" it is
  Technically, *Hausdorff dimension* from measure theory is used
- **Infinite detail**: They exhibit intricate detail at arbitrarily small scales

Fractals: The geometry of roughness



Koch curve



Koch snowflake

Example: Koch snowflake
The join of three *Koch curves* along the sides of an equilateral triangle

The Koch curve is the limit of the construction:

1. divide a segment into three equal parts
2. draw an equilateral triangle that has the middle segment from the previous step as its base and points outward
3. remove the base of the triangle

The snowflake has a perimeter of infinite length; its area is $\frac{8}{5}A$ with A the area of the initial triangle

The Hausdorff dimension of the Koch curve is $d = \frac{\ln 4}{\ln 3} \approx 1.26\ldots$

Fractals: The geometry of roughness

Example: The Mandelbrot set
Perhaps the most famous fractal, generated by iterating a simple complex number equation. Let $z, c \in \mathbb{C}$,

$$f_c(z) = z^2 + c$$

the Mandelbrot set is defined as

$$M = \{c \mid |f_c(0)|, |f_c(f_c(0))|, |f_c(f_c(f_c(0)))|, \cdots \text{ does not diverge to } \infty\}$$

Its border has topological dimension 1 because it is a curve, however its Hausdorff dimension is 2

Fractals: The geometry of roughness

Connections and applications:

- **Chaos theory**:
  Strange attractors in chaotic systems often display a fractal structure
- **Nature**:
  Modelling coastlines, river networks, tree branching, snowflakes, blood vessel systems
- **Computer graphics**:
  Generating realistic landscapes and special effects
- **Signal processing, medicine, finance**:
  Analysing complex data and patterns

Unveiling hidden geometry: Fractal geometry provided mathematicians and scientists with the tools to describe, analyse, and even generate the complex, irregular patterns prevalent in both natural phenomena and abstract mathematical systems, fundamentally expanding our understanding of geometry itself

Major unsolved problems

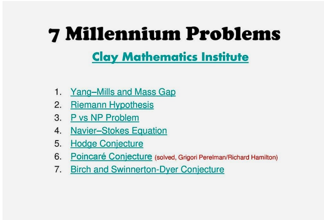
The unfinished symphony of mathematics:

- Despite the monumental achievements of the 20$^{\text{th}}$ century — solving long-standing conjectures (FLT, Hilbert's 10$^{\text{th}}$) and developing vast new theories — the mathematical landscape is still filled with profound unanswered questions
- These open problems often act as powerful driving forces for new research, pushing the boundaries of existing theories and leading to unexpected connections

Major unsolved problems

The Millennium Prize Problems

- The Clay Mathematics Institute (CMI) in 2000 announced seven "Millennium Prize Problems", offering a $1 million prize for the first correct solution to each
- They represent some of the most challenging and fundamental open problems in mathematics



The Clay Millennium Prize problems

- So far, only one has been solved (Poincaré conjecture)
  Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere

Major unsolved problems

A glimpse at remaining challenges:

- **P vs NP problem**: (Already discussed) Is finding a solution as easy as checking one? Crucial for computer science
- **Riemann hypothesis**: (Already discussed) The fine-grained distribution of prime numbers
- **Yang-Mills existence and mass gap**: To establish a rigorous foundation for quantum field theories used in particle physics
- **Navier-Stokes existence and smoothness**: To understand the existence and properties of solutions to equations describing fluid flow (crucial for weather, aerodynamics)
- **Birch and Swinnerton-Dyer conjecture**: Deep problem in number theory about the number of rational points on elliptic curves
- **Hodge conjecture**: A problem in algebraic geometry relating topology to algebraic curves

The enduring frontiers

The endless quest: These unsolved problems are not deficits but rather invitations — proof that mathematics is a vibrant, expanding universe of ideas, continuously challenging our intellect and inspiring new generations of mathematicians to explore its boundless frontiers

20th century mathematics

A century of revolution, unification, and endless discovery

- **Foundational scrutiny & abstraction** (early 20th century): Began with a crisis of foundations, leading to unprecedented rigour, axiomatic systems (**ZFC**, **Bourbaki**), and the rise of highly abstract structures (general topology, functional analysis, modern algebra & geometry)
- **Interconnections & grand problem solving** (mid-20th century): Mathematics matured into a deeply interconnected web. New tools from one field solved problems in another (e.g., **Fermat's last theorem** via elliptic curves and modular forms; **Hilbert's 10th problem** via computability). Geometry provided the language for physics (**general relativity**, but also **Hilbert spaces**)

20^{th} century mathematics

A century of revolution, unification, and endless discovery

- **Computation, complexity & new frontiers**
  (late 20^{th}/early 21^{st} centuries):
  The advent of the **Turing machine** and **computability theory** revolutionised mathematics and laid the groundwork for the digital age. Fields like **chaos theory** and **fractal geometry** explored complexity

20^{th} century mathematics

Profound impact: The 20^{th} century transformed mathematics from a discipline primarily focused on physical models into a vast, abstract, and deeply unified science. Its intellectual output has not only expanded human knowledge but has profoundly shaped modern technology, medicine, finance, and our fundamental understanding of the universe.

The future of mathematics: As we step into the 21^{st} century, mathematics continues its dynamic evolution. Driven by new technologies, massive data, and enduring unsolved mysteries like the **Riemann hypothesis** and **P vs NP**, mathematicians are constantly forging new connections, developing new tools, and exploring an ever-expanding universe of ideas. The journey of discovery is far from over

References

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- *Morris Kline*, Mathematical Thought from Ancient to Modern Times, Oxford University Press (1972)
- *Piergiorgio Odifreddi*, La matematica del Novecento: Dagli insiemi alla complessità, Einaudi (2000)

However, the most recent aspects come from the direct knowledge of the instructor

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