MATHEMATICAL LOGIC — ASSIGNMENT FOUR

(1) It is a known fact that if $\vdash_{\mathsf{PA}} A$ then $\vdash_{\mathsf{HA}} (A)^N$, in which $\vdash_{\mathsf{PA}} A$ denotes provability in Peano arithmetic, $\vdash_{\mathsf{HA}}$ denotes provability in Heyting arithmetic, and $(\_)^N$ is the Gödel-Gentzen translation.

Prove that Peano arithmetic is consistent if and only if Heyting arithmetic is consistent.

(2) Prove the fixed point lemma:

> Let Ξ be a theory in which every partial recursive function is representable and let A be a formula such that $\mathsf{FV}(A) = \{ y \}$.
> Then there is a formula δ_A such that $\mathsf{FV}(\delta_A) = \emptyset$ and $\vdash_{\Xi} \delta_A = A[\ulcorner \delta_A \urcorner / y]$.

(3) Show that $\neg A \vee \neg\neg A$, which is an instance of the Law of Excluded Middle thus valid in classical logic, is not provable in intuitionistic logic.

(Hint: Remember that the usual topology on $\mathbb{R}$ is a Heyting algebra.)

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

Date: June 3[rd], 2026.