

MATHEMATICAL LOGIC — ASSIGNMENT TWO

- (1) Prove  $\vdash (\forall x. P \vee \neg P) \wedge \neg \forall x. \neg P \supset \exists x. P$ .

$$\frac{\frac{\frac{[(\forall x. P \vee \neg P) \wedge \neg \forall x. \neg P]^2}{\neg \forall x. \neg P} \wedge E_1}{\frac{(\exists x. P) \vee \neg \exists x. P \text{ lem}}{\exists x. P} \vee E_1} \quad \frac{\frac{\frac{[P]^3}{\exists x. P} \exists I \quad [\neg \exists x. P]^1}{\perp} \neg E}{\frac{\perp}{\neg P} \neg I^3}{\forall x. \neg P} \forall I}{\perp} \neg E}{\exists x. P} \perp E}{\frac{\exists x. P}{(\forall x. P \vee \neg P) \wedge \neg \forall x. \neg P \supset \exists x. P} \supset I^2} \vee E^1$$

- (2) Prove the Downward Löwenheim-Skolem Theorem:

Let  $T$  be a theory on the signature  $\Sigma$  with just one sort. If  $T$  has an infinite model of cardinality  $\alpha \geq |\Sigma|$  then  $T$  has a model of cardinality  $\max(|\Sigma|, \aleph_0)$ .

**This is Theorem 13.5 in the slides.**

- (3) Show that for every partial order  $\langle P; \leq_P \rangle$  there exists a total order  $\langle P; \leq_T \rangle$  such that  $\leq_P \subseteq \leq_T$ .

(Hint: a set  $S$  is finite if and only if there is a function  $f: \{n \mid 0 \leq n < |S|\} \rightarrow S$  bijective.)

Let  $\mathbb{P} = \langle P; \leq_P \rangle$  be a partial order. Fix the signature

$$\Sigma = \langle \{S\}; \{c_x: S \mid x \in P\}; \{\leq: S \times S, =: S \times S\} \rangle$$

and let  $T$  be the theory containing the following axioms:

$\forall x. x \leq x$	reflexivity
$\forall x, y. x \leq y \wedge y \leq x \supset x = y$	anti-symmetry
$\forall x, y, z. x \leq y \wedge y \leq z \supset x \leq z$	transitivity
$\forall x, y. x \leq y \vee y \leq x$	totality
$c_x \leq c_y$ for all $x \leq_P y$	embedding of $\mathbb{P}$
$\neg c_x = c_y$ for all $x, y \in P, x \neq y$	embedding of $\mathbb{P}$

Let  $F$  be a finite subset of  $T$ . Define  $C = \{c_x \mid x \text{ occurs in } F\}$ : since  $F$  is finite, so is  $C$ , thus there is  $f: \{n \mid 0 \leq n < |C|\} \rightarrow C$  bijective.

Hence, the natural numbers with the usual order  $\leq_{\mathbb{N}}$  are a model for  $F$  posing  $\llbracket c_x \rrbracket = f^{-1}(x)$  for every  $x \in C$ , and  $\llbracket c_x \rrbracket = 42$  for every  $x \notin C$ .

Thus  $T$  has a model  $\mathfrak{M}$  by the Compactness Theorem.

Define  $\leq_T$  on  $P \times P$  as:  $x \leq_T y$  if and only if  $\llbracket c_x \leq c_y \rrbracket$  holds in  $\mathfrak{M}$ .

Then  $\leq_P \subseteq \leq_T$  since we have embedded  $\leq_P$  in  $T$ ; also,  $\leq_T$  is a total order because  $\mathfrak{M}$  makes  $T$  valid.

Thus  $\langle P; \leq_T \rangle$  is a total order extending  $\mathbb{P}$ , as required.