MATHEMATICAL LOGIC — ASSIGNMENT ONE

(1) Prove in natural deduction $\vdash ((A \supset B) \supset A) \supset A$.

$$\dfrac{\dfrac{\dfrac{}{A \vee \neg A}\,\text{lem} \qquad [A]^1 \qquad \dfrac{[(A \supset B) \supset A]^2 \qquad \dfrac{\dfrac{\dfrac{[\neg A]^1 \quad [A]^3}{\bot}\,\neg\mathrm{E}}{B}\,\bot\mathrm{E}}{A \supset B}\,\supset\mathrm{I}^3}{A}\,\supset\mathrm{E}}{A}\,\vee\mathrm{E}^1}{((A \supset B) \supset A) \supset A}\,\supset\mathrm{I}^2$$

(2) Show that for each x and y in a lattice, $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$.
This is Proposition 5.11 in the slides.

(3) Consider the following inference rule

$$\dfrac{\begin{matrix}[\neg A]\\ \vdots \\ \bot\end{matrix}}{A}\,\bot_c$$

Show that classical propositional logic is equivalent to the calculus without the Law of Excluded Middle plus the $\bot_c$ rule.

The $\bot_c$ rule can be simulated in classical natural deduction. Having a proof $\pi \colon \neg A \vdash \bot$, the following construction simulates $\bot_c$:

$$\dfrac{\dfrac{}{A \vee \neg A}\,\text{lem} \qquad [A]^1 \qquad \dfrac{\begin{matrix}[\neg A]^1\\ \vdots\ \pi \\ \bot\end{matrix}}{A}\,\bot\mathrm{E}}{A}\,\vee\mathrm{E}^1$$

Conversely, the Law of Excluded Middle is proved by

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(A \vee \neg A)]^1 \qquad \dfrac{[A]^2}{A \vee \neg A}\,\vee\mathrm{I}_1}{\bot}\,\neg\mathrm{E}}{\neg A}\,\neg\mathrm{I}^2 \qquad \dfrac{\dfrac{[\neg(A \vee \neg A)]^1 \qquad \dfrac{[\neg A]^3}{A \vee \neg A}\,\vee\mathrm{I}_2}{\bot}\,\neg\mathrm{E}}{\neg\neg A}\,\neg\mathrm{I}^3}{\bot}\,\neg\mathrm{E}}{A \vee \neg A}\,\bot_c^1$$

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