MATHEMATICAL LOGIC — ASSIGNMENT FOUR

(1) Define the notion of representable relation in Peano arithmetic.

This is Definition 27.2 in the slides.

(2) Prove that in every Heyting algebra, for every x, y, z, $x \leq y \supset z$ if and only if $x \wedge y \leq z$.

This is Proposition 23.2 in the slides.

(3) Assuming that Peano arithmetic is consistent, show that there is a sentence in Heyting arithmetic which is not provable and whose negation is not provable either.

(Hint: use Gödel's First Incompleteness Theorem for Peano arithmetic. Beware that provability in Heyting arithmetic differs from provability in Peano arithmetic.)

Let $\Gamma \vdash_{\mathsf{PA}} A$ denote that A is provable from Γ in Peano arithmetic, and $\Gamma \vdash_{\mathsf{HA}} A$ denote that A is provable from Γ in Heyting arithmetic.

Observe that Peano and Heyting arithmetic have the same axioms, but the former operates in classical logic, while the latter in intuitionistic logic.

The provability predicate T_{HA} for Heyting arithmetic is defined exactly as T, the provability predicate for Peano arithmetic we have seen in the course, using the same coding function g but dropping the Law of Excluded Middle from the coding of derivations.

By the Fixed Point Lemma, there is a sentence G_{HA} such that

$$\vdash_{\mathsf{PA}} G_{\mathsf{HA}} = \neg \mathsf{T}_{\mathsf{HA}} [\ulcorner G_{\mathsf{HA}} \urcorner / y] \ .$$

Suppose $\vdash_{\mathsf{HA}} G_{\mathsf{HA}}$. Since every intuitionistic proof is classical, $\vdash_{\mathsf{PA}} G_{\mathsf{HA}}$. Hence $\vdash_{\mathsf{PA}} \neg \mathsf{T}_{\mathsf{HA}} [\ulcorner G_{\mathsf{HA}} \urcorner / y]$. Thus $\nvdash_{\mathsf{HA}} G_{\mathsf{HA}}$ because T_{HA} represents provability in Heyting arithmetic, contradiction.

Suppose $\vdash_{\mathsf{HA}} \neg G_{\mathsf{HA}}$, thus $\vdash_{\mathsf{PA}} \neg G_{\mathsf{HA}}$ since every intuitionistic derivation is also classical, so $\vdash_{\mathsf{PA}} \mathsf{T}_{\mathsf{HA}} [\ulcorner G_{\mathsf{HA}} \urcorner / y]$ by definition of G_{HA}, hence there is $n \in \mathbb{N}$ such that $g^{-1}(n)$ is the code of a proof $\pi \colon \vdash_{\mathsf{HA}} G_{\mathsf{HA}}$. As before, this implies $\pi \colon \vdash_{\mathsf{PA}} G_{\mathsf{HA}}$, thus Peano arithmetic would be non consistent, contradiction.

Hence there is a sentence, G_{HA}, in Heyting arithmetic such that $\nvdash_{\mathsf{HA}} G_{\mathsf{HA}}$ and $\nvdash_{\mathsf{HA}} \neg G_{\mathsf{HA}}$.

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