MATHEMATICAL LOGIC — ASSIGNMENT TWO

(1) Prove $\vdash (\forall x.\, A) \vee B \supset \forall x.\, A \vee B$ where $x \notin \mathsf{FV}(B)$. Show a counterexample when $x \in \mathsf{FV}(B)$.

$$\dfrac{\dfrac{[(\forall x.\, A) \vee B]^1 \quad \dfrac{\dfrac{[\forall x.\, A]^2}{A}\,\forall E}{A \vee B}\,\vee I_1 \quad \dfrac{[B]^2}{A \vee B}\,\vee I_2}{\dfrac{A \vee B}{\forall x.\, A \vee B}\,\forall I}\,\vee E^2}{(\forall x.\, A) \vee B \supset \forall x.\, A \vee B}\,\supset I^1$$

Let $A \equiv \bot$, $B \equiv$ 'x is even' in arithmetic, and suppose to take an evaluation of variables in which $x \mapsto 2$. Then $[\![\forall x.\, A \vee B]\!] = \bot$ while $[\![(\forall x.\, A) \vee B]\!] = \top$.

(2) State and prove the Compactness Theorem.

This is Theorem 12.1 in the slides.

(3) In the definition of Σ-structure it is required that all sorts are interpreted in non-empty sets. Show that this requirement is necessary since the Soundness Theorem becomes invalid if the requirement is not met.

Consider a single-sorted signature with equality, and a Σ-structure $\mathfrak{E}$ interpreting the sort in the empty set. By the following proof

$$\dfrac{\dfrac{\dfrac{}{\forall x.\, x = x}\,\text{refl}}{x = x}\,\forall E}{\exists x.\, x = x}\,\exists I$$

assuming the Soundness Theorem to hold, it follows that $[\![\exists x.\, x = x]\!] = 1$ in every model. So, in every model there is an element $[\![x]\!]$ such that $[\![x = x]\!] = 1$. But this is false in $\mathfrak{E}$, since there is no element in the interpretation of the sort.

Hence, $\exists x.\, x = x$ is provable but not valid in every model, contradicting the validity of the Soundness Theorem.

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