MATHEMATICAL LOGIC — ASSIGNMENT ONE

(1) Prove in natural deduction $\vdash \neg(A \vee B) = \neg A \wedge \neg B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(A\vee B)]^1 \quad \dfrac{[A]^2}{A\vee B}{\scriptstyle \vee\mathrm{I}_1}}{\bot}{\scriptstyle \neg\mathrm{E}}}{\neg A}{\scriptstyle \neg\mathrm{I}^2} \qquad \dfrac{\dfrac{[\neg(A\vee B)]^1 \quad \dfrac{[B]^3}{A\vee B}{\scriptstyle \vee\mathrm{I}_2}}{\bot}{\scriptstyle \neg\mathrm{E}}}{\neg B}{\scriptstyle \neg\mathrm{I}^3}}{\neg A\wedge\neg B}{\scriptstyle \wedge\mathrm{I}}}{\neg(A\vee B)\supset\neg A\wedge\neg B}{\scriptstyle \supset\mathrm{I}^1}$$

$$\dfrac{\dfrac{\dfrac{[A\vee B]^1 \quad \dfrac{[A]^2 \quad \dfrac{[\neg A\wedge\neg B]^3}{\neg A}{\scriptstyle \wedge\mathrm{E}_1}}{\bot}{\scriptstyle \neg\mathrm{E}} \quad \dfrac{[B]^2 \quad \dfrac{[\neg A\wedge\neg B]^3}{\neg B}{\scriptstyle \wedge\mathrm{E}_2}}{\bot}{\scriptstyle \neg\mathrm{E}}}{\bot}{\scriptstyle \vee\mathrm{E}^2}}{\neg(A\vee B)}{\scriptstyle \neg\mathrm{I}^1}}{\neg A\wedge\neg B\supset\neg(A\vee B)}{\scriptstyle \supset\mathrm{I}^3}$$

(2) Show that in a bounded lattice $\langle S; \leq\rangle$, $\bigvee S = \top$ and $\bigwedge S = \bot$.

This is Proposition 5.7 in the slides.

(3) Show that every non-trivial Boolean algebra which is also a total order, is isomorphic to $\langle\{0, 1\}; \leq\rangle$, the Boolean algebra of truth-tables. Here, non-trivial means that $\bot \neq \top$.

In a Boolean algebra every element x has a unique complement $\neg x$ such that $x \wedge \neg x = \bot$ and $x \vee \neg x = \top$.

Fix a non-trivial Boolean algebra which is also a total order. Then for every element x one of the following options holds

- $x \leq \neg x$, thus $x \wedge \neg x = x$, so $x = \bot$;
- $\neg x \leq x$, thus $x \vee \neg x = x$, so $x = \top$.

Hence every element x is either $\top$ or $\bot$, so the support of the algebra is $\{\bot, \top\}$, and $\bot < \top$ by definition of $\bot$ and $\top$. Clearly, this structure is isomorphic to the algebra supporting truth-tables.

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