Strong normalisation in the full simple theory of types

Categorical Logic and Constructive Mathematics

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Full simple theory of types

Assume the separated variables convention and suppose to fix a set of *type constants*. The full simple theory of types is then

$$\frac{}{x : T} \qquad \frac{t : T_2 \quad x : T_1}{\lambda x : T_1\, t : T_1 \to T_2} \qquad \frac{t : T_1 \to T \quad t_1 : T_1}{t \cdot t_1 : T}$$

$$\frac{t_1 : T_1 \quad t_2 : T_2}{(t_1, t_2) : T_1 \times T_2} \qquad \frac{t : T_1 \times T_2}{t \cdot \pi_1 : T_1} \qquad \frac{t : T_1 \times T_2}{t \cdot \pi_2 : T_2}$$

$$\frac{t : T_1}{\mathrm{inl}_{T_2}\, t : T_1 + T_2} \qquad \frac{t : T_2}{\mathrm{inr}_{T_1}\, t : T_1 + T_2}$$

$$\frac{t : T_1 + T_2 \quad a : T \quad b : T}{t \cdot \delta(x_0 : T_1.\, a, x_1 : T_2.\, b) : T} \qquad \frac{t : \mathbf{0}}{t \cdot \phi_T : T}$$

where x, x_0, x_1 are variables, and λ and δ bind x and x_0, x_1 respectively.

Elimination-oriented syntax

Eliminator: an element R such that $t \cdot R$ is a term.

May be a term a, π_1, π_2, ϕ_T, or $\delta(x_0 : A.\, a, x_1 : B.\, b)$.

Vector notation: $\overrightarrow{r}$ denotes a list $r_1, \ldots, r_n$ of terms; $\overrightarrow{R}$ denotes a list $R_1, \ldots, R_n$ of eliminators.

Vector application: $t \cdot \overrightarrow{R}$ is t when $n = 0$, and $(t \cdot \overrightarrow{S}) \cdot R_n$ with $\overrightarrow{S} = R_1, \ldots, R_{n-1}$ when $n > 0$.

Vector relations: $\overrightarrow{R} \in X$ is a shorthand for $R_1 \in X, \ldots, R_n \in X$, where the π_1, π_2, and ϕ_T eliminators vacuously satisfy membership, and $\delta(x_0.\, a, x_1.\, b) \in X$ means $a \in X$ and $b \in X$. Other relations, e.g., $x \notin \mathrm{FV}(\overrightarrow{R})$, are similarly defined.

Reductions

$\rhd_1$, *one-step reduction*, is the minimal congruence containing $\blacktriangleright$.

$\rhd$, *reduction*, is the reflexive and transitive closure of $\rhd_1$.

Computation contractions:

- $(\lambda x{:}A\, b)\cdot a \blacktriangleright b[x/a]$
- $(a,b)\cdot\pi_1 \blacktriangleright a \qquad (a,b)\cdot\pi_2 \blacktriangleright b$
- $(\mathsf{inl}_B\, a)\cdot\delta(x{:}A.\,b, y{:}B.\,c) \blacktriangleright b[x/a]$ with $x \notin \mathsf{FV}(a)$
  $(\mathsf{inr}_A\, a)\cdot\delta(x{:}A.\,b, y{:}B.\,c) \blacktriangleright c[y/a]$ with $y \notin \mathsf{FV}(a)$
  renaming the bounded variables if needed

Uniqueness contractions:

- $\lambda x{:}A\, a\cdot x \blacktriangleright a$ where $x \notin \mathsf{FV}(a)$
- $(a\cdot\pi_1, a\cdot\pi_2) \blacktriangleright a$
- $a\cdot\delta(x{:}A.\,\mathsf{inl}_B\, x, y{:}B.\,\mathsf{inr}_A\, y) \blacktriangleright a$
- $a\cdot\phi_{\mathbf{0}} \blacktriangleright a$

Reductions

Permutation contractions:

- $a \cdot \phi_{A\times B} \blacktriangleright (a\cdot\phi_A, a\cdot\phi_B)$
- $a\cdot\phi_{A\to B} \blacktriangleright \lambda x : A\, a\cdot\phi_B$ with $x \notin \mathrm{FV}(a)$
- $a\cdot\phi_{A+B} \blacktriangleright \mathrm{inl}_B(a\cdot\phi_A)$
- if $r\cdot\delta(x:A.\,a, y:B.\,b) : C\times D$ then

$$r\cdot\delta(x:A.\,a, y:B.\,b) \blacktriangleright (r\cdot\delta(x:A.\,a\cdot\pi_1, y:B.\,b\cdot\pi_1),\\ r\cdot\delta(x:A.\,a\cdot\pi_2, y:B.\,b\cdot\pi_2))$$

- if $r\cdot\delta(x:A.\,a, y:B.\,b) : C\to D$ and $z\notin \{x,y\}\cup \mathrm{FV}(r\cdot\delta(x:A.\,a,y:B.\,b))$ then

$$r\cdot\delta(x:A.\,a, y:B.\,b) \blacktriangleright \lambda z : C\, r\cdot\delta(x:A.\,a\cdot z, y:B.\,b\cdot z)$$

- if $x\notin \mathrm{FV}(c)$ and $y \notin \mathrm{FV}(d)$ then

$$(r\cdot\delta(x:A.\,a, y:B.\,b))\cdot\delta(z:C.\,c, w:D.\,d)\\ \blacktriangleright r\cdot\delta(x:A.\,a\cdot\delta(z:C.\,c,w:D.\,d), y:B.\,b\cdot\delta(z:C.\,c,w:D.\,d))$$

Strongly normalisable

A term t is *strongly normalisable* (sn) if (α) there is a bound $m \in \mathbb{N}$ to the length of every reduction sequence starting from t.

Equivalently, a term t is strongly normalisable when (β) every term r such that $t \rhd_1 r$ is strongly normalisable.

A theory of types is strongly normalisable when every term which can be typed in it, is strongly normalisable.

[(α) and (β) are induction principles]

Naive strong normalisation

Lemma 2.1

- $\lambda x : A\, a$ *is sn iff* a *is sn*
- (a, b) *is sn iff* a *and* b *are sn*
- $\mathsf{inl}_T\, a$ *is sn iff* a *is sn*
- $\mathsf{inr}_T\, a$ *is sn iff* a *is sn*
- $\pi_1\, a$ *is sn iff* a *is sn*
- $\pi_2\, a$ *is sn iff* a *is sn*
- $a \cdot \phi_T$ *is sn iff* a *is sn*

Lemma 2.2

If $a[x/b]$ *is sn, so is* a*.*

Naive strong normalisation

Lemma 2.3

- *If* $a : A \to T$ *is sn and* $x : A$ *is a variable such that* $x \notin \mathrm{FV}(a)$*, then* $a \cdot x$ *is sn*
- *If* $a \cdot b$ *is sn, so are* a *and* b

Lemma 2.4

- *If* t*,* f*, and* g *are sn, and* t *is neither a* δ*-term nor an injection, then* $t \cdot \delta(u.f, v.g)$ *is sn.*
- *If* $e_1 \cdot \delta(x.e_2, y.e_3)$ *is sn then* e_i *is sn*

The proofs of these lemmata are straightforward inductions on one-step reductions or the length of reductions, possibly augmented by an external induction on the types of whole terms.

[from now on, these lemmata are taken for granted]

The set **SN**

SN is the minimal set of typable terms closed by the rules in the following.

We claim that:

- if $t \in \mathbf{SN}$ then t is sn
- if $t : T$ is term, then $t \in \mathbf{SN}$

Strong normalisation is an immediate consequence.

The set **SN**

$$\frac{\vec{R} \in \mathbf{SN}}{x\vec{R} \in \mathbf{SN}}\,(V_1) \qquad \frac{x\vec{R} \in \mathbf{SN} \quad a : T \in \mathbf{SN} \quad b : T \in \mathbf{SN}}{x\vec{R}\delta(y.a, z.b) \in \mathbf{SN}}\,(V_2)$$

$$\frac{a \in \mathbf{SN}}{\lambda x : T\, a \in \mathbf{SN}}\,(\lambda) \quad \frac{a \in \mathbf{SN} \quad b \in \mathbf{SN}}{(a, b) \in \mathbf{SN}}\,(p) \quad \frac{a \in \mathbf{SN}}{\mathrm{inl}_T\, a \in \mathbf{SN}}\,(i_1) \quad \frac{a \in \mathbf{SN}}{\mathrm{inr}_T\, a \in \mathbf{SN}}\,(i_2)$$

$$\frac{(a[x/b])\vec{D} \in \mathbf{SN} \quad b \in \mathbf{SN}}{(\lambda x : T\, a)b\vec{D} \in \mathbf{SN}}\,(\beta)$$

$$\frac{a\vec{D} \in \mathbf{SN} \quad b \in \mathbf{SN}}{(a, b)\pi_1\vec{D} \in \mathbf{SN}}\,(\pi_1) \qquad \frac{a \in \mathbf{SN} \quad b\vec{D} \in \mathbf{SN}}{(a, b)\pi_2\vec{D} \in \mathbf{SN}}\,(\pi_2)$$

- In the rules, $\vec{R}$ is a list of eliminators containing no δ's and ϕ's. Oppositely $\vec{D}$ is a generic, unconstrained list of eliminators.
- In (V_2) the type T is required to be either atomic or a sum.
- x is a variable.

The set **SN**

$$\frac{(b[x/a])\overrightarrow{D} \in \mathbf{SN} \quad a \in \mathbf{SN} \quad b\overrightarrow{D} \in \mathbf{SN} \quad c\overrightarrow{D} \in \mathbf{SN}}{(\mathsf{inl}_B\, a)\delta(x.b, y.c)\overrightarrow{D} \in \mathbf{SN}}\ (\delta_1)$$

$$\frac{(c[x/a])\overrightarrow{D} \in \mathbf{SN} \quad a \in \mathbf{SN} \quad b\overrightarrow{D} \in \mathbf{SN} \quad c\overrightarrow{D} \in \mathbf{SN}}{(\mathsf{inr}_A\, a)\delta(x.b, y.c)\overrightarrow{D} \in \mathbf{SN}}\ (\delta_2)$$

$$\frac{(a \cdot \delta(y.b \cdot \pi_1, z.c \cdot \pi_1), a \cdot \delta(y.b \cdot \pi_2, z.c \cdot \pi_2))\,\overrightarrow{D} \in \mathbf{SN}}{a \cdot \delta(y.b, z.c)\overrightarrow{D} \in \mathbf{SN}}\ (\delta_\times)$$

$$\frac{(\lambda u : A\, a \cdot \delta(y.b \cdot u, z.c \cdot u))\,\overrightarrow{D} \in \mathbf{SN}}{a \cdot \delta(y.b, z.c)\overrightarrow{D} \in \mathbf{SN}}\ (\delta_\to)$$

$$\frac{a \cdot \delta(y.b \cdot \delta(u.f, v.g), z.c \cdot \delta(u.f, v.g))\overrightarrow{D} \in \mathbf{SN}}{a \cdot \delta(y.b, z.c)\delta(u.f, v.g)\overrightarrow{D} \in \mathbf{SN}}\ (\delta_+)$$

The set **SN**

$$\frac{a \in \mathbf{SN}}{a \cdot \phi_T \in \mathbf{SN}}\,(\phi_a) \qquad \frac{(a \cdot \phi_A, a \cdot \phi_B)\,\overrightarrow{D} \in \mathbf{SN}}{a\phi_{A\times B}\overrightarrow{D} \in \mathbf{SN}}\,(\phi_\times)$$

$$\frac{(\lambda x : A\, a \cdot \phi_B)\,\overrightarrow{D} \in \mathbf{SN}}{a\phi_{A\to B}\overrightarrow{D} \in \mathbf{SN}}\,(\phi_\to) \qquad \frac{(\mathrm{inl}_B\,(a \cdot \phi_A))\,\overrightarrow{D} \in \mathbf{SN}}{a\phi_{A+B}\overrightarrow{D} \in \mathbf{SN}}\,(\phi_+)$$

- In $(\delta_\times)$ the type of $a \cdot \delta(y.b, z.c)$ is a product $A \times B$.
- In $(\delta_\to)$ the type of $a \cdot \delta(y.b, z.c)$ is a function space $A \to B$.
- In (ϕ_a) the type T is required to be atomic.
- In all the rules, the same constraints as for contractions apply on free and bounded variables.

Combinatorics of δ

Definition 4.1

To cope with nested δ-eliminators and their permutations, it is useful to define Δ_n and ∇_n:

$$\Delta_n(a, b_1, c_1, \ldots, b_n, c_n) = a \cdot \delta(x_1.\, b_1, y_1.\, c_1) \cdots \delta(x_n.\, b_n, y_n.\, c_n)$$

and

$$\begin{aligned}\nabla_n(a, b_1, c_1, \ldots, b_n, c_n) = a \cdot \delta(x_1.\, &\nabla_{n-1}(b_1, b_2, c_2, \ldots, b_n, c_n),\\ &y_1.\, \nabla_{n-1}(c_1, b_2, c_2, \ldots, b_n, c_n))\end{aligned}$$

with $\Delta_0(a) = a = \nabla_0(a)$.

Lemma 4.2

For every n and for every a, b_1, ..., b_n, c_1, ..., c_n sn terms, $\Delta_n(a, b_1, c_1, \ldots, b_n, c_n)$ is sn iff $\nabla_n(a, b_1, c_1, \ldots, b_n, c_n)$ is so.

Proof. (i)

Suppose $\Delta_n(\ldots)$ is sn. Since it reduces to $\nabla_n(\ldots)$, $\nabla_n(\ldots)$ is sn, too.

Conversely, for every n and for every a, b_1, ..., b_n, c_1, ..., c_n sn, if $\nabla_n(a, b_1, c_1, \ldots, b_n, c_n) : T$ is sn and $\Delta_n(a, b_1, c_1, \ldots, b_n, c_n) \rhd_1 r$ then r is sn. The proof is by induction on

$$\left(n, T, \operatorname{maxred}(a) + \sum_{j=1}^{n} \operatorname{maxred}(b_j) + \sum_{j=1}^{n} \operatorname{maxred}(c_j)\right)$$

lexicographically ordered. ↪

[maxred(t) is the length of the longest reduction sequence starting from t]

Combinatorics of δ

$\hookrightarrow$ Proof. (ii)

Considering the possible one-step reductions:

- $r = \Delta_n(a', b_1, c_1, \ldots, b_n, c_n)$ and $a \rhd_1 a'$. Since n and T are unchanged in r, but $\text{maxred}(a') < \text{maxred}(a)$, r is sn by IH. Likewise for $b_i \rhd_1 b'$ or $c_i \rhd_1 c'$.
- $r = \Delta_{n-1}(b_1[x/a'], b_2, c_2, \ldots, b_n, c_n)$ and $a = \text{inl}_{C_1}\, a'$. Since

$$\nabla_n(a, b_1, c_1, \ldots, b_n, c_n) \rhd_1 \nabla_{n-1}(b_1[x/a'], b_2, c_2, \ldots, b_n, c_n)$$

  and the lhs side is sn, so is the rhs side, thus the subterm $b_1[x/a']$ is sn. Then r is sn by IH. Similarly when $a = \text{inr}_{B_1}\, a'$.
- $r = \Delta_{n-1}(a, b_1, c_1, \ldots, b_{i-1}, c_{i-1}, b_{i+1}, c_{i+1}, \ldots, b_n, c_n)$ and $b_i = \text{inl}_{C_{i+1}}\, x_i$, $c_i = \text{inr}_{B_{i+1}}\, y_i$. Thus r is sn by IH. $\hookrightarrow$

Combinatorics of δ

↪ Proof. (iii)

- $T = C \times D$ and

$$r = \big(\Delta_n(a, b_1, c_1, \ldots, b_{n-1}, c_{n-1}, b_n \cdot \pi_1, c_n \cdot \pi_1) : C,$$
$$\Delta_n(a, b_1, c_1, \ldots, b_{n-1}, c_{n-1}, b_n \cdot \pi_2, c_n \cdot \pi_2) : D\big) \ .$$

  Since $b_n \cdot \pi_1$, $b_n \cdot \pi_2$, $c_n \cdot \pi_1$, and $c_n \cdot \pi_2$ are sn, $\Delta_n(a, b_1, c_1, \ldots, b_{n-1}, c_{n-1}, b_n \cdot \pi_1, c_n \cdot \pi_1)$ and $\Delta_n(a, b_1, c_1, \ldots, b_{n-1}, c_{n-1}, b_n \cdot \pi_2, c_n \cdot \pi_2)$ are sn by IH, since n is the same, but their types are proper subtypes of T. Hence r is sn.

- $T = C \to D$ and $r = \lambda w : C \, \Delta_n(a, b_1, c_1, \ldots, b_{n-1}, c_{n-1}, b_n \cdot w, c_n \cdot w)$. Since $b_n \cdot w$ and $c_n \cdot w$ are sn

$$\Delta_n(a, b_1, c_1, \ldots, b_{n-1}, c_{n-1}, b_n \cdot w, c_n \cdot w)$$

  is sn by IH since its type D is a proper subtype of T. Thus r is sn. ↪

Combinatorics of δ

↪ Proof. (iv)

- r is the result of a permutation contraction on a sum type:

$$\begin{aligned} r = \Delta_{n-1}(a, & b_1, c_1, \ldots, b_{i-1}, c_{i-1}, \\ & b_i \cdot \delta(x_{i+1}.\, b_{i+1}, y_{i+1}.\, c_{i+1}), \\ & c_i \cdot \delta(x_{i+1}.\, b_{i+1}, y_{i+1}.\, c_{i+1}), \\ & b_{i+2}, c_{i+2}, \ldots, b_n, c_n) \end{aligned}$$

and $1 \leq i < n$. Considering $R = \nabla_{n-i}(b_i, b_{i+1}, c_{i+1}, \ldots, b_n, c_n)$, this term is a subterm of $\nabla_n(a, b_1, c_1, \ldots, b_n, c_n)$, which is sn by hypothesis, thus R is sn, too. Hence $\Delta_{n-i}(b_i, b_{i+1}, c_{i+1}, \ldots, b_n, c_n)$ is sn by IH, so its head $b_i \cdot \delta(x_{i+1}.\, b_{i+1}, y_{i+1}.\, c_{i+1})$ is sn. Likewise, $c_i \cdot \delta(x_{i+1}.\, b_{i+1}, y_{i+1}.\, c_{i+1})$ is sn. Then r is sn by IH. □

$t \in \mathbf{SN}$ implies t sn

Theorem 4.3

$\mathbf{SN} \subseteq \{t \mid t \textit{ is sn}\}$.

Proof. (i)

By induction on the generation of **SN**: if $t \in \mathbf{SN}$ is the conclusion of a rule, it has to be shown that t is sn, assuming the premises of the rule are sn.

- (V_1), (V_2), (λ), (p), (i_1), (i_2), and (ϕ_a) are easy instances of the naive lemmata combined with the IH. $\hookrightarrow$

$t \in \mathbf{SN}$ implies t sn

$\hookrightarrow$ Proof. (ii)

- (β) Since $(a[x/b])\overrightarrow{D} \in \mathbf{SN}$ and $b \in \mathbf{SN}$, then $(a[x/b])\overrightarrow{D}$ and b are sn by IH, thus a, $a[x/b]$ and each eliminator in $\overrightarrow{D}$ are sn.
  Then $(\lambda x : T\, a)b\overrightarrow{D}$ is sn by side induction on $|\overrightarrow{D}|$.

  If $|\overrightarrow{D}| = 0$, consider $(\lambda x : T\, a) \cdot b \rhd_1 r$. Then r is sn by inner induction on $\text{maxred}(a) + \text{maxred}(b)$: if $r = (\lambda x : T\, a') \cdot b$ and $a \rhd_1 a'$, or $r = (\lambda x : T\, a) \cdot b'$ and $b \rhd_1 b'$, then r is sn by inner IH; if $r = a[x/b]$ because of β-contraction or η-contraction then r is sn as above.

  Hence, $(\lambda x : T\, a)b$ is sn since it reduces in one step to sn terms only. $\hookrightarrow$

$t \in \mathbf{SN}$ implies t sn

$\hookrightarrow$ Proof. (iii)

If $\overrightarrow{D} = \overrightarrow{E}\,G$ then $(\lambda x : T\,a)b\overrightarrow{E}$ is sn by side IH on $\overrightarrow{E}$.
Let $((\lambda x : T\,a)b\overrightarrow{E}) \cdot G \rhd_1 r$. Then r is sn by inner induction on $\text{maxred}\Big((\lambda x : T\,a)b\overrightarrow{E}\Big) + \text{maxred}(G)$: if r is obtained reducing a component in the application , then r is sn by inner IH.
If G is a term, $G = \pi_1$, or $G = \pi_2$, no other reductions are possible. If $G = \phi_C$ then $(\lambda x : T\,a)b\overrightarrow{D}$ is sn, thus also r is sn.

So let G be a δ-eliminator. If $\overrightarrow{E} = \overrightarrow{F}\delta(y.F_1, z.F_2)$ and $r = (\lambda y : T\,a)b\overrightarrow{F}\delta(y.F_1\,G, z.F_2\,G)$ then

$$(a[x/b])\overrightarrow{D} \rhd_1 (a[x/b])\overrightarrow{F}\delta(y.F_1\,G, z.F_2\,G) \ ,$$

in which the lhs is sn, thus it is so also the rhs, hence $\delta(y.F_1\,G, z.F_2\,G)$ is sn. Then r is sn by side IH.
If the last eliminator in $\overrightarrow{E}$, if any, is not a δ-eliminator, then $(\lambda x : T\,a)b\overrightarrow{D}$ is sn by the naive lemma on δ. Thus r is sn. $\hookrightarrow$

$t \in \mathbf{SN}$ implies t sn

↪ Proof. (iv)

- (π_1) and (π_2) are analogous to the (β) case.
- (δ_1) is very similar to (β). However, one more case has to be considered: when $\overrightarrow{D} = G$, G is a δ-eliminator, and $r = (\mathrm{inl}_B\, a)\delta(x.\, b \cdot G, y.\, c \cdot G)$, then $b\overrightarrow{D} = b \cdot G$ and $c\overrightarrow{D} = c \cdot G$. Thus $b \cdot G$ and $c \cdot G$ are sn by main IH, so $\delta(x.\, b \cdot G, y.\, c \cdot G)$ is sn. Then r is sn by side IH.
- (δ_2) Symmetrical to the case (δ_1).
- $(\delta_\times)$, $(\delta_\rightarrow)$, $(\phi_\times)$, $(\phi_\varnothing)$, and (ϕ_+) are, in turn, analogous to (δ_1). ↪

$t \in \mathbf{SN}$ implies t sn

↪ Proof. (v)

- (δ_+) It is convenient to write the conclusion as $\Delta_n(a, b_1, c_1, \ldots, b_n, c_n)\overrightarrow{D} \in \mathbf{SN}$ where a is not a δ-term and $\overrightarrow{D}$ is empty or its first element is not a δ-eliminator. Then the premise of the rule becomes, for some $1 \le i < n$,

$$\begin{aligned} P = \Delta_{n-1}(a, & b_1, c_1, \ldots, b_{i-1}, c_{i-1}, \\ & b_i \cdot \delta(x_{i+1}.\, b_{i+1}, y_{i+1}.\, c_{i+1}), \\ & c_i \cdot \delta(x_{i+1}.\, b_{i+1}, y_{i+1}.\, c_{i+1}), \\ & b_{i+2}, c_{i+2}, \ldots, b_n, c_n)\overrightarrow{D} \in \mathbf{SN} \end{aligned}$$

  Thus P is sn by IH. Since $P \rhd \nabla_n(a, b_1, c_1, \ldots, b_n, c_n)\overrightarrow{D}$, $\nabla_n(a, b_1, c_1, \ldots, b_n, c_n)\overrightarrow{D}$ is sn. Thus $\Delta_n(a, b_1, c_1, \ldots, b_n, c_n)$ is sn by the previous lemma. Hence $\Delta_n(a, b_1, c_1, \ldots, b_n, c_n)\overrightarrow{D}$ is sn by induction on $|\overrightarrow{D}|$ as in the previous cases. □

SN and term formation

Theorem 5.1

The following facts hold, assuming all the terms are typable:

1. *if* x *is variable then* $x \in \mathbf{SN}$;
2. *if* $a \in \mathbf{SN}$ *then* $\lambda x : T\, a \in \mathbf{SN}$;
3. *if* $a \in \mathbf{SN}$ *and* $b \in \mathbf{SN}$ *then* $(a, b) \in \mathbf{SN}$;
4. *if* $a \in \mathbf{SN}$ *then* $\mathrm{inl}_T\, a \in \mathbf{SN}$ *and* $\mathrm{inr}_T\, a \in \mathbf{SN}$;
5. *if* $T = A \to B$ *and* $e : T \in \mathbf{SN}$ *then* $e \cdot a \in \mathbf{SN}$ *for all* $a \in \mathbf{SN}$;
6. *if* $T = A \times B$ *and* $e : T \in \mathbf{SN}$ *then* $e \cdot \pi_1 \in \mathbf{SN}$ *and* $e \cdot \pi_2 \in \mathbf{SN}$;
7. *if* $T = A + B$ *and* $e : T \in \mathbf{SN}$ *then* $e \cdot \delta(x : A. a, y : B. b) \in \mathbf{SN}$ *for all* $a : C \in \mathbf{SN}$ *and* $b : C \in \mathbf{SN}$;
8. *if* $a \in \mathbf{SN}$ *then* $a \cdot \phi_T \in \mathbf{SN}$;
9. *if* $x : T$ *is a variable and* $e : A \in \mathbf{SN}$ *then* $e[x/a] \in \mathbf{SN}$ *for all* $a : T \in \mathbf{SN}$.

SN and term formation

Proof. (i)

Properties (1)–(4), are instances of (V_1), (λ), (p), (i_1), and (i_2).

Property (8) is proved by induction on T:

- T atomic. Then $a \cdot \phi_T \in \mathbf{SN}$ by (ϕ_a).
- $T = A \times B$. Then $a \cdot \phi_A \in \mathbf{SN}$ and $a \cdot \phi_B \in \mathbf{SN}$ by IH on A and B. Thus $(a \cdot \phi_A, a \cdot \phi_B) \in \mathbf{SN}$ by (p), hence $a \cdot \phi_T \in \mathbf{SN}$ by $(\phi_\times)$.
- $T = A \to B$. Then $a \cdot \phi_B \in \mathbf{SN}$ by IH on B. Let $x : A \notin \mathrm{FV}(a)$. Thus $\lambda x : A\, a \cdot \phi_B \in \mathbf{SN}$ by (λ), hence $a \cdot \phi_T \in \mathbf{SN}$ by $(\phi_\to)$.
- $T = A + B$. Then $a \cdot \phi_A \in \mathbf{SN}$ by IH on A. Thus $\mathrm{inl}_B(a \cdot \phi_A) \in \mathbf{SN}$ by (i_1), hence $a \cdot \phi_T \in \mathbf{SN}$ by (ϕ_+). ↪

SN and term formation

↪ Proof. (ii)

Property (6) has to show that $e \cdot \pi_1 \in \mathbf{SN}$ and $e \cdot \pi_2 \in \mathbf{SN}$ from $e : T = A \times B \in \mathbf{SN}$. The proof is by induction on the generation of $e \in \mathbf{SN}$:

- $e = x\overrightarrow{R}$ by (V_1), so $\overrightarrow{R} \in \mathbf{SN}$. Hence $e \cdot \pi_1 \in \mathbf{SN}$ and $e \cdot \pi_2 \in \mathbf{SN}$ by (V_1).
- $e = (a, b)$ by (p), so $a \in \mathbf{SN}$ and $b \in \mathbf{SN}$. Thus $e \cdot \pi_1 \in \mathbf{SN}$ by (π_1) and $e \cdot \pi_2 \in \mathbf{SN}$ by (π_2).
- $e = (\lambda x : C\, b)c\overrightarrow{D}$ by (β), so $b[x/c]\overrightarrow{D} \in \mathbf{SN}$ and $c \in \mathbf{SN}$. Hence $b[x/c]\overrightarrow{D} \cdot \pi_1 \in \mathbf{SN}$ and $b[x/c]\overrightarrow{D} \cdot \pi_2 \in \mathbf{SN}$ by IH, thus $e \cdot \pi_1 \in \mathbf{SN}$ and $e \cdot \pi_2 \in \mathbf{SN}$ by (β).

  The cases for the rules (π_1), (π_2), (δ_1), (δ_2), $(\delta_\times)$, $(\delta_\to)$, (δ_+), $(\phi_\times)$, $(\phi_\to)$, and (ϕ_+) are analogous. ↪

SN and term formation

$\hookrightarrow$ Proof. (iii)

Two technical properties are needed to continue the proof:

- (T1) if $a \in \mathbf{SN}$ and $x \notin \mathrm{FV}(a)$ then $a[y/x] \in \mathbf{SN}$. This property holds by an immediate induction on the generation of $a \in \mathbf{SN}$.
- (T2) if $e : A \to B \in \mathbf{SN}$ and $w \notin \mathrm{FV}(e)$ then $e \cdot w \in \mathbf{SN}$.

Property (T2) is proved by induction on the generation of $e \in \mathbf{SN}$, observing that $w \in \mathbf{SN}$ by (V_1) on no premises:

- $e = z\overrightarrow{R}$ by (V_1), so $\overrightarrow{R} \in \mathbf{SN}$, thus $e \cdot w \in \mathbf{SN}$ by (V_1).
- $e = \lambda y : A\, b$ by (λ), so $b \in \mathbf{SN}$. Since $b[y/w] \in \mathbf{SN}$ by Property (T1), $e \cdot w \in \mathbf{SN}$ by (β).
- $e = (\lambda x : C\, b)c\overrightarrow{D}$ by (β), so $b[x/c]\overrightarrow{D} \in \mathbf{SN}$ and $c \in \mathbf{SN}$. Then $b[x/c]\overrightarrow{D} \cdot w \in \mathbf{SN}$ by IH, thus $e \cdot w \in \mathbf{SN}$ by (β).

  The cases for the rules (π_1), (π_2), (δ_1), (δ_2), $(\delta_\times)$, $(\delta_\to)$, (δ_+), $(\phi_\times)$, $(\phi_\to)$, and (ϕ_+) are similar. $\hookrightarrow$

SN and term formation

↪ Proof. (iv)

Properties (5), (7) and (9) are proved together by induction on (T, e), where T is ordered by the subtype relation, e is ordered by the generation of **SN**; as usual, pairs are lexicographically ordered.

To show (5), consider the possible ways in which $e : T \to A \in \mathbf{SN}$:

- $e = x\overrightarrow{R}$ by (V_1), so $\overrightarrow{R} \in \mathbf{SN}$. Since $a \in \mathbf{SN}$ by hypothesis, $e \cdot a \in \mathbf{SN}$ by (V_1).
- $e = \lambda x : A\, b$ by (λ), so $b \in \mathbf{SN}$. Since $a \in \mathbf{SN}$ by hypothesis and A is a proper subtype of T, $b[x/a] \in \mathbf{SN}$ by IH, so $e \cdot a \in \mathbf{SN}$ by (β).
- $e = (\lambda x : C\, b)c\overrightarrow{D}$ by (β), so $b[x/c]\overrightarrow{D} \in \mathbf{SN}$ and $c \in \mathbf{SN}$. Since $a : A \in \mathbf{SN}$ and A is a proper subtype of $T = A \to B$, $b[x/c]\overrightarrow{D} \cdot a \in \mathbf{SN}$ by IH, thus $e \cdot a \in \mathbf{SN}$ by (β).

  The cases for the rules (π_1), (π_2), (δ_1), (δ_2), $(\delta_\times)$, $(\delta_\to)$, (δ_+), $(\phi_\times)$, $(\phi_\to)$, and (ϕ_+) are similar. ↪

SN and term formation

↪ Proof. (v)

Property (7) is proved using the same technique. It has to be shown that $e \cdot \delta(x.a, y.b) : C \in \mathbf{SN}$ from $e : T = A + B$ for all $a, b \in \mathbf{SN}$:

- $e = z\overrightarrow{R}$ by (V_1). By side induction on the type C:
  - if C is either atomic or a sum, then $e \cdot \delta(x.a, y.b) \in \mathbf{SN}$ by (V_2).
  - if $C = D \times E$ then $a \cdot \pi_1, a \cdot \pi_2, b \cdot \pi_1, b \cdot \pi_2 \in \mathbf{SN}$ by Property (6), thus $e \cdot \delta(x.a \cdot \pi_1, y.b \cdot \pi_1) \in \mathbf{SN}$ and $e \cdot \delta(x.a \cdot \pi_2, y.b \cdot \pi_2) \in \mathbf{SN}$ by side IH on D and E, respectively.
    Hence $(e \cdot \delta(x.a \cdot \pi_1, y.b \cdot \pi_1), e \cdot \delta(x.a \cdot \pi_2, y.b \cdot \pi_2)) \in \mathbf{SN}$ by (p), then $e \cdot \delta(x.a, y.b) \in \mathbf{SN}$ by $(\delta_\times)$.
  - if $C = D \to E$, let $w : D$ be a variable such that $w \notin \{x, y\} \cup \mathrm{FV}(a) \cup \mathrm{FV}(b) \cup \mathrm{FV}(e)$. Then $a \cdot w \in \mathbf{SN}$ and $b \cdot w \in \mathbf{SN}$ by Property (T2), thus $e \cdot \delta(x.a \cdot w, y.b \cdot w) \in \mathbf{SN}$ by side IH on D.
    Hence $\lambda w : C\, e \cdot \delta(x.a \cdot w, y.b \cdot w) \in \mathbf{SN}$ by (λ), then $e \cdot \delta(x.a, y.b) \in \mathbf{SN}$ by $(\delta_\to)$, concluding the side induction. ↪

SN and term formation

↪ Proof. (vi)

- $e = z\overrightarrow{R}\delta(u.f, v.g)$ by (V_2), so $z\overrightarrow{R} \in \mathbf{SN}$, $f : T \in \mathbf{SN}$, and $g : T \in \mathbf{SN}$. Thus $f \cdot \delta(x.a, y.b) \in \mathbf{SN}$ and $g \cdot \delta(x.a, y.b) \in \mathbf{SN}$ by IH on f and g.
  Then $z\overrightarrow{R}\delta(u.f \cdot \delta(x.a, y.b), v.g \cdot \delta(x.a, y.b)) \in \mathbf{SN}$ by (V_1), hence $e \cdot \delta(x.a, y.b) \in \mathbf{SN}$ by (δ_+).
- $e = \mathrm{inl}_A\, c$ by (i_1), so $c \in \mathbf{SN}$. Since $e : T = A + B$ and $c : A$, $a[x/e] \in \mathbf{SN}$ by IH on A. Hence $e \cdot \delta(x.a, y.b) \in \mathbf{SN}$ by (δ_1).
  The case $e = \mathrm{inr}_B\, c$ by (i_2) is symmetrical.
- $e = (\lambda w : C\, d)\, c\overrightarrow{D}$ by (β), so $d[w/c]\overrightarrow{D} \in \mathbf{SN}$ and $c \in \mathbf{SN}$. Since $e : T$, also $d[w/c]\overrightarrow{D} : T$, thus $d[w/c]\overrightarrow{D}\delta(x.a, y.b) \in \mathbf{SN}$ by IH. Hence $e \cdot \delta(x.a, y.b) \in \mathbf{SN}$ by (β).

  The cases for the rules (π_1), (π_2), (δ_1), (δ_2), $(\delta_\times)$, $(\delta_\to)$, (δ_+), $(\phi_\times)$, $(\phi_\to)$, and (ϕ_+) are analogous. ↪

SN and term formation

$\hookrightarrow$ Proof. (vii)

Property (9) is similar, but additional care has to be taken when the variable to substitute is x in $x\overrightarrow{R}$. It has to be shown that $e[x/a] \in \mathbf{SN}$ from $e, a \in \mathbf{SN}$.

- $e = w\overrightarrow{R}$ by (V_1), so $r[x/a] \in \mathbf{SN}$ by IH on each term $r \in \overrightarrow{R}$.
  If $x \neq w$ then $e[x/a] = w(\overrightarrow{R}[x/a]) \in \mathbf{SN}$ by (V_1).
  Thus assume $x = w$. Then $(x\overrightarrow{R})[x/a] \in \mathbf{SN}$ by side induction on $|\overrightarrow{R}|$: if $\overrightarrow{R}$ is empty then $(x\overrightarrow{R})[x/a] = a \in \mathbf{SN}$ by hypothesis. Otherwise $\overrightarrow{R} = \overrightarrow{S}\,G$ and $(x\overrightarrow{S})[x/a] \in \mathbf{SN}$ by side IH.
  If $G = \pi_1$ or $G = \pi_2$ then $(x\overrightarrow{R})[x/a] = (x\overrightarrow{S})[x/a] \cdot G \in \mathbf{SN}$ by Property (6).
  Otherwise $G : B$ is a term. Let $z : B$ be a variable such that $z \notin \mathrm{FV}(a) \cup \mathrm{FV}(x\overrightarrow{R})$. Then $(x\overrightarrow{R})[x/a] = ((x\overrightarrow{S})[x/a] \cdot z)[z/G[x/a]]$.
  Since $(x\overrightarrow{S})[x/a] \cdot z \in \mathbf{SN}$ by Property (T2), $G[x/a] \in \mathbf{SN}$ as already seen, and B is a proper subtype of T, then $(x\overrightarrow{R})[x/a] \in \mathbf{SN}$ by IH. $\hookrightarrow$

SN and term formation

↪ Proof. (viii)

- $e = w\overrightarrow{R}\delta(y.b, z.c)$ by (V_2), so $(w\overrightarrow{R})[x/a] \in \mathbf{SN}$, $b[x/a] \in \mathbf{SN}$, and $c[x/a] \in \mathbf{SN}$ by IH. It is safe to assume $y, z \notin \{x\} \cup \mathrm{FV}(a)$.
  If $x \neq w$ then $e[x/a] = (w\overrightarrow{R})[x/a] \cdot \delta(y.b[x/a], z.c[x/a]) \in \mathbf{SN}$ by (V_2).
  Thus assume $x = w$ and let $x\overrightarrow{R} : B + C$. If $\overrightarrow{R}$ is inhabited then $B + C$ is a proper subtype of T, the type of a and x. Hence

$$e[x/a] = ((x\overrightarrow{R})[x/a]) \cdot \delta(y.b[x/a], z.c[x/a]) \in \mathbf{SN}$$

  by IH on $B + C$. ↪

↪ Proof. (ix)

Therefore, let $\overrightarrow{R}$ be empty, so $e[x/a] = a \cdot \delta(y.\, b[x/a], z.\, c[x/a])$ whose type is atomic or a sum. Then $e[x/a] \in \mathbf{SN}$ is proved by showing that $a \in \mathbf{SN}$ implies $a \cdot \delta(y.\, b[x/a], z.\, c[x/a])$ by side induction on the generation of $a \in \mathbf{SN}$:

- $a = u\overrightarrow{S} \in \mathbf{SN}$ by (V_1). Then $e[x/a] = u\overrightarrow{S} \cdot \delta(y.\, b[x/a], z.\, c[x/a]) \in \mathbf{SN}$ by (V_2) since its type is atomic or a sum.
- $a = u\overrightarrow{S}\delta(v_1.\, f, v_2.\, g)$ by (V_2), so $u\overrightarrow{S}, f, g \in \mathbf{SN}$. Thus

$$f \cdot \delta(y.\, b[x/a], z.\, c[x/a]) \in \mathbf{SN}$$
$$g \cdot \delta(y.\, b[x/a], z.\, c[x/a]) \in \mathbf{SN}$$

  by side IH on f and g. Then

$$u\overrightarrow{S}\delta(v_1.\, f \cdot \delta(y.\, b[x/a], z.\, c[x/a]),\; v_2.\, g \cdot \delta(y.\, b[x/a], z.\, c[x/a])) \in \mathbf{SN}$$

  by (V_2), hence $e[x/a] \in \mathbf{SN}$ by (δ_+). ↪

SN and term formation

$\hookrightarrow$ Proof. (x)

- $a = \mathrm{inl}_C\, a'$ by (i_1), so $a' : B \in \mathbf{SN}$. Since $T = B + C$, $(b[x/a])[y/a'] \in \mathbf{SN}$ by main IH on B. Hence $e[x/a] \in \mathbf{SN}$ by (δ_1).
  The case $a = \mathrm{inr}_B\, a'$ by (i_2) is symmetrical.
- $a = (\lambda u : E\, f)\, g\, \overrightarrow{D}$ by (β), so $f[u/g]\overrightarrow{D} \in \mathbf{SN}$ and $g \in \mathbf{SN}$. Thus

$$f[u/g]\overrightarrow{D} \cdot \delta(y.\, b[x/a], z.\, c[x/a]) \in \mathbf{SN}$$

  by side IH on $f[u/g]\overrightarrow{D}$. Hence $e[x/a] \in \mathbf{SN}$ by (β).

  The cases for all the other suitable rules, (π_1), (π_2), (δ_1), (δ_2), $(\delta_\times)$, $(\delta_\to)$, (δ_+), and (ϕ_+), are analogous. This concludes the (V_2) case. $\hookrightarrow$

SN and term formation

↪ Proof. (xi)

- $e = \lambda y : B\, b$ by (λ), so $b \in \mathbf{SN}$. It is safe to assume $y \notin \{x\} \cup \mathrm{FV}(a)$. Thus $b[x/a] \in \mathbf{SN}$ by IH on b. Hence $e[x/a] = \lambda y : B\, b[x/a] \in \mathbf{SN}$ by (λ).

  The cases for the rules (p), (i_1), (i_2), and (ϕ_a) are analogous.

- $e = (\lambda y : B\, c)\, b \overrightarrow{D}$ by (β), so $c[y/b]\overrightarrow{D} \in \mathbf{SN}$ and $b \in \mathbf{SN}$. It is safe to assume $y \notin \{x\} \cup \mathrm{FV}(a)$. Then $(c[y/b]\overrightarrow{D})[x/a] \in \mathbf{SN}$ and $b[x/a] \in \mathbf{SN}$ by IH. Since

$$\begin{aligned}
&(c[y/b]\overrightarrow{D})[x/a] \\
&= (c[y/b][x/a])(\overrightarrow{D}[x/a]) \\
&= ((c[x/a])[y/b[x/a]])(\overrightarrow{D}[x/a]) \ ,
\end{aligned}$$

  $e[x/a] = (\lambda y : B\, c[x/a])\, (b[x/a])(\overrightarrow{D}[x/a]) \in \mathbf{SN}$ by (β).

  The cases for the rules (δ_1) and (δ_2) are analogous. ↪

SN and term formation

↪ Proof. (xii)

- $e = (b, c)\pi_1\overrightarrow{D}$ by (π_1), so $b\overrightarrow{D} \in \mathbf{SN}$ and $c \in \mathbf{SN}$. Then $(b\overrightarrow{D})[x/a] = (b[x/a])(\overrightarrow{D}[x/a]) \in \mathbf{SN}$ and $c[x/a] \in \mathbf{SN}$ by IH. Hence $(b[x/a], c[x/a])\pi_1(\overrightarrow{D}[x/a])) = e[x/a] \in \mathbf{SN}$ by (π_1).

  The cases for the rules (π_2), $(\phi_\times)$, $(\phi_\to)$, and (ϕ_+) are similar. ↪

SN and term formation

↪ Proof. (xiii)

- $e = b \cdot \delta(y.c, z.d)\overrightarrow{D}$ by $(\delta_\times)$, so

$$\big(b \cdot \delta(y.c \cdot \pi_1, z.d \cdot \pi_1), b \cdot \delta(y.c \cdot \pi_2, z.d \cdot \pi_2)\big)\overrightarrow{D} \in \mathbf{SN} \ .$$

It is safe to assume $y, z \notin \{x\} \cup \mathrm{FV}(a)$. Thus, by IH,

$$\begin{aligned} &\Big(\big(b \cdot \delta(y.c \cdot \pi_1, z.d \cdot \pi_1), b \cdot \delta(y.c \cdot \pi_2, z.d \cdot \pi_2)\big)\overrightarrow{D}\Big)[x/a] \\ &= \big(b[x/a] \cdot \delta(y.c[x/a] \cdot \pi_1, z.d[x/a] \cdot \pi_1), \\ &\quad b[x/a] \cdot \delta(y.c[x/a] \cdot \pi_2, z.d[x/a] \cdot \pi_2)\big)(\overrightarrow{D}[x/a]) \in \mathbf{SN} \ . \end{aligned}$$

Hence $b[x/a] \cdot \delta(y.c[x/a], z.d[x/a])(\overrightarrow{D}[x/a]) = e[x/a] \in \mathbf{SN}$ by $(\delta_\times)$.

The cases for the rules $(\delta_\to)$ and (δ_+) are analogous. □

References

This proof follows the general guidelines of F. Joachimksi and R. Matthes, *Short Proofs of Normalization*, Archive for Mathematical Logic 42, 59–87 (2003).

The main differences are

- slightly more general permutation rules
- $\perp$ ($\mathbf{0}$ type) and $\wedge$ ($\times$ type constructor) have also been considered
- all the uniqueness rules have been included in the proof
- a mistake in the published proof, which has been later corrected by the authors, is avoided by an alternative proof pattern
- the proof has the same general structure, but many local aspects use different approaches, e.g., the combinatorics of permutations via Δ and ∇
- more details (!!!)

The end