MATHEMATICAL LOGIC — ASSIGNMENT FOUR

(1) Define the Gödel-Gentzen translation.

This is Definition 22.2 in the slides.

(2) Prove that there is a non-standard model for Peano arithmetic.

This is Theorem 26.2 in the slides.

(3) Let $P = \{ \ulcorner\phi\urcorner \colon \vdash_{\mathsf{PA}} \phi \text{ and } \mathsf{FV}(\phi) = \emptyset \}$. Show that P is not recursive.

Suppose P is recursive. Then it is representable in Peano arithmetic: there is a formula Q with $\mathsf{FV}(Q) = \{ y \}$ such that

- if $\ulcorner\phi\urcorner \in P$ then $\vdash_{\mathsf{PA}} Q\,[\ulcorner\phi\urcorner/y]$;
- if $\ulcorner\phi\urcorner \notin P$ then $\vdash_{\mathsf{PA}} \neg Q\,[\ulcorner\phi\urcorner/y]$.

Then there is a sentence G such that $\vdash_{\mathsf{PA}} G = \neg Q\,[\ulcorner G\urcorner/y]$ by the Fixed Point Lemma.

If $\ulcorner G\urcorner \in P$ then $\vdash_{\mathsf{PA}} G$ by definition of P, thus $\vdash_{\mathsf{PA}} \neg Q\,[\ulcorner G\urcorner/y]$ by definition of G, hence $\ulcorner G\urcorner \notin P$ by definition of Q: contradiction.

If $\ulcorner G\urcorner \notin P$ then $\vdash_{\mathsf{PA}} \neg Q\,[\ulcorner G\urcorner/y]$ by definition of Q, thus $\vdash_{\mathsf{PA}} G$ by definition of G, hence $\ulcorner G\urcorner \in P$ by definition of P: contradiction.

Therefore, P cannot be recursive.

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