MATHEMATICAL LOGIC — ASSIGNMENT TWO

(1) Prove $\vdash A[t/x] = \exists x.\, x = t \wedge A$ in natural deduction assuming $x \notin \mathsf{FV}(t)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\, x = x}\,\text{refl}}{t = t}\,\forall\text{E} \qquad [A[t/x]]^1}{t = t \wedge A[t/x]}\,\wedge\text{I}}{\exists x.\, x = t \wedge A}\,\exists\text{I}}{A[t/x] \supset \exists x.\, x = t \wedge A}\,\supset\text{I}^1$$

$$\dfrac{\dfrac{[\exists x.\, x = t \wedge A]^1 \qquad \dfrac{\dfrac{[x = t \wedge A]^2}{x = t}\,\wedge\text{E}_1 \qquad \dfrac{[x = t \wedge A]^2}{A}\,\wedge\text{E}_2}{A[t/x]}\,\text{subst}}{A[t/x]}\,\exists\text{E}^2}{(\exists x.\, x = t \wedge A) \supset A[t/x]}\,\supset\text{I}^1$$

(2) Show that, for any set of formulæ Γ and any formula A,
- $\Gamma \cup \{ \neg A \}$ is not consistent if and only if $\Gamma \vdash A$;
- $\Gamma \cup \{ A \}$ is not consistent if and only if $\Gamma \vdash \neg A$.

This is Proposition 10.3 in the slides.

(3) Show that every first-order theory T on the signature

$$\langle S; \emptyset; \{ \leq : S \times S \} \rangle$$

having as models all the finite total orders, has necessarily an infinite model.

Let $C = \{ c_i : i \in \mathbb{N} \}$ be a set of constants extending the given signature.

Observe how $=$ can defined by the formula $\forall x.\, \forall y.\, x \leq y \wedge y \leq x$.

Let $\Xi = \{ \neg c_i = c_j : i, j \in \mathbb{N} \wedge i \neq j \}$ be a set of axioms.

Consider a finite $F \subseteq T \cup \Xi$. Hence, there is finite number of axioms in F from Ξ, thus, in particular, there is finite number m of constants in C appearing in F.

Then F has $\langle \{0, \ldots, m\} ; \leq \rangle$, the usual order on naturals restricted to the first m numbers, as a model.

Thus, by the Compactness Theorem, $T \cup \Xi$ has a model $\mathcal{M}$, and all the c_i's are interpreted in distinct elements, thus $\mathcal{M}$ is infinite.

Hence, $\mathcal{M}$ is a model for T and it is infinite.

Date: April 10^{th}, 2024.