MATHEMATICAL LOGIC — ASSIGNMENT ONE

(1) Prove $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \supset (B \supset C)]^1 \quad [A]^2}{B \supset C}{\supset}\mathrm{E} \qquad \dfrac{[A \supset B]^3 \quad [A]^2}{B}{\supset}\mathrm{E}}{C}{\supset}\mathrm{E}}{A \supset C}{\supset}\mathrm{I}^2}{(A \supset B) \supset (A \supset C)}{\supset}\mathrm{I}^3}{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}{\supset}\mathrm{I}^1$$

(2) Show that in any distributive lattice for all x, y, and z,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \ .$$

This is Proposition 5.12 in the slides.

(3) Show that if $\vdash A \supset B$ then $[\![A]\!] \leq [\![B]\!]$ in every Boolean algebra.

From $\vdash A \supset B$ it holds $[\![A \supset B]\!] = \top$ in every Boolean algebra by the Soundness Theorem. Since $[\![A \supset B]\!] = \neg[\![A]\!] \vee [\![B]\!]$, it follows

$$\begin{aligned}
&[\![A]\!] \\
&= [\![A]\!] \wedge \top \\
&= [\![A]\!] \wedge (\neg[\![A]\!] \vee [\![B]\!]) \\
&= ([\![A]\!] \wedge \neg[\![A]\!]) \vee ([\![A]\!] \wedge [\![B]\!]) \\
&= \bot \vee ([\![A]\!] \wedge [\![B]\!]) \\
&= [\![A]\!] \wedge [\![B]\!]
\end{aligned}$$

By definition of meet, $[\![A]\!] \wedge [\![B]\!] \leq [\![B]\!]$, hence $[\![A]\!] \leq [\![B]\!]$.

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