MATHEMATICAL LOGIC — ASSIGNMENT FOUR

(1) Consider the term in the Simple Theory of Types

$$\lambda x\colon ((A \to 0) \to 0) \to 0.\, \lambda y\colon A.\, x\,(\lambda z\colon A \to 0.\, z\, y)\ .$$

Calculate its type and write the corresponding proof in natural deduction according to the propositions-as-types interpretation.

The type X of $\lambda x\colon ((A \to 0) \to 0) \to 0.\, \lambda y\colon A.\, x\,(\lambda z\colon A \to 0.\, z\, y)$ is $(((A \to 0) \to 0) \to 0) \to Y$ where Y is the type of $\lambda y\colon A.\, x\,(\lambda z\colon A \to 0.\, z\, y)$, so $Y \equiv A \to W$ where W is the type of $x\,(\lambda z\colon A \to 0.\, z\, y)$. Thus, $W \equiv 0$ because the type of $\lambda z\colon A \to 0.\, z\, y$ is $(A \to 0) \to 0$. Therefore, the required type is $(((A \to 0) \to 0) \to 0) \to (A \to 0)$.

Applying the propositions-as-types interpretation:

$$\dfrac{\dfrac{\dfrac{[\neg\neg\neg A]^1 \qquad \dfrac{\dfrac{[\neg A]^2 \quad [A]^3}{\bot}\,{\supset}\mathrm{E}}{\neg\neg A}\,{\supset}\mathrm{I}^2}{\bot}\,{\supset}\mathrm{E}}{\neg A}\,{\supset}\mathrm{I}^3}{\neg\neg\neg A \supset \neg A}\,{\supset}\mathrm{I}^1$$

in which negation $\neg P$ is treated as an abbreviation for $P \supset \bot$.

(2) State, without proving, Gödel's second incompleteness theorem.

This is Theorem 29.5 in the slides.

(3) In the language of Peano arithmetic, let T be the theory containing all the true closed formulae in a standard model. Show that the set $g(T)$, where g is Gödel's coding of formulae, is not recursive, i.e., there is no computable way to decide whether any closed formula is true on naturals.

[Hint: The Weak Incompleteness Theorem may be useful.]

Clearly, T is consistent, because it has a model. If ϕ is true in every model and $\mathsf{FV}(\phi) = \{x_1, \ldots, x_n\}$ then the sentence $\psi \equiv \forall x_1, \ldots, x_n.\, \phi$ is true in every model, in particular the standard one, so it occurs in T.

Hence, $T \vdash \phi$ because $T \vdash \psi$ by the axiom rule, followed by an n-length sequence of instances of the $\forall$-elimination rule.

Therefore, T is sound and complete, thus it is not effective by the Weak Incompleteness Theorem (31.1 in the slides), that is, the set T is not enumerable by a recursive function.

Suppose $g(T)$ is recursive: then there is a recursive function h such that $h(x) = 1$ if and only if $x = g(\phi)$ and $\phi \in T$. Then

$$f(0) = \mu x.\, h(x) = 1$$
$$f(n+1) = \mu x.\, h(x) = 1 \land f(n) < x$$

Date: June 14^{th}, 2023.

is a recursive function enumerating $g(T)$, thus $f \circ g$ is a recursive function enumerating T, which is impossible.