

EXERCISES IN NATURAL DEDUCTION

- (1) $\vdash A \supset (B \supset A)$
- (2) $\vdash (A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
- (3) $\vdash A = \neg\neg A$
- (4) $\vdash \neg A = \neg\neg\neg A$
- (5) $\vdash \neg(A \vee B) = \neg A \wedge \neg B$
- (6) $\vdash (A \supset B) \vee (B \supset A)$
- (7) $\vdash \neg A \supset (A \supset B)$
- (8) $\vdash ((A \supset B) \supset A) \supset A$
- (9) $\vdash \neg\neg(A \supset B) = \neg\neg A \supset \neg\neg B$
- (10) $\vdash A \supset B = \neg B \supset \neg A$
- (11) $\vdash A \vee B = \neg A \supset B$
- (12) $\vdash A \supset B = \neg(A \wedge \neg B)$
- (13) $\vdash A \wedge B \supset \neg(A \supset \neg B)$
- (14) $\vdash (A \vee B) \vee C = A \vee (B \vee C)$
- (15) $\vdash \neg\neg(A \wedge B) = \neg\neg A \wedge \neg\neg B$
- (16) $\vdash (A \wedge B) \wedge C = A \wedge (B \wedge C)$
- (17) $\vdash A \vee B = B \vee A$
- (18) $\vdash A \wedge B = B \wedge A$
- (19) $\vdash A \vee A = A$
- (20) $\vdash A \wedge A = A$
- (21) $\vdash A \vee (A \wedge B) = A$
- (22) $\vdash A \wedge (A \vee B) = A$
- (23) $\vdash A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
- (24) $\vdash A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
- (25) $\vdash \neg(A \vee B) = \neg A \wedge \neg B$
- (26) $\vdash \neg(A \wedge B) = \neg A \vee \neg B$
- (27) $\vdash A \wedge B = \neg(\neg A \vee \neg B)$
- (28) $\vdash A \vee B = \neg(\neg A \wedge \neg B)$
- (29) $\vdash A \supset B = \neg A \vee B$
- (30) $\vdash (A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
- (31) $\vdash A \supset (B \supset C) = B \supset (A \supset C)$
- (32) $\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$
- (33) $\vdash A \supset (B \supset C) = A \wedge B \supset C$
- (34) $\vdash \neg(A \wedge \neg A)$
- (35) $\vdash \neg(A \supset B) = A \wedge \neg B$
- (36) $\vdash (A \wedge B \supset C) \supset (A \supset C) \vee (B \supset C)$
- (37) $\vdash (A \supset \neg A) \supset \neg A$
- (38) $\vdash (A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)$
- (39) $\vdash (A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C)$
- (40) $\vdash (A \supset B) \wedge (A \supset C) \supset (A \supset B \wedge C)$
- (41) $\vdash (A \supset B) \wedge (B \supset C) \supset (A \supset C)$
- (42) $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

- (43) $\vdash \forall x. B \supset A = B \supset \forall x. A$ with $x \notin \text{FV}(B)$
- (44) $\vdash \forall x. A \supset B = (\exists x. A) \supset B$ with $x \notin \text{FV}(B)$
- (45) $\vdash \neg\neg\forall x. A \supset \forall x. \neg\neg A$
- (46) $\vdash A \wedge \forall x. B = \forall x. A \wedge B$ with $x \notin \text{FV}(A)$
- (47) $\vdash A \wedge \exists x. B = \exists x. A \wedge B$ with $x \notin \text{FV}(A)$
- (48) $\vdash A \vee \exists x. B = \exists x. A \vee B$ with $x \notin \text{FV}(A)$
- (49) $\vdash (\forall x. P \vee \neg P) \wedge \neg\forall x. \neg P \supset \exists x. P$
- (50) $\vdash \forall x. A \wedge B = (\forall x. A) \wedge (\forall x. B)$
- (51) $\vdash (\exists x. A \wedge B) \supset (\exists x. A) \wedge (\exists x. B)$
- (52) $\vdash (\exists x. A \wedge B) \supset (\exists x. A) \vee (\exists x. B)$
- (53) $\vdash \exists x. A \vee B = (\exists x. A) \vee (\exists x. B)$
- (54) $\vdash (\forall x. A) \supset \exists x. A$
- (55) $\vdash (\exists x. \forall y. A) \supset \forall y. \exists x. A$
- (56) $\vdash (\exists x. A) = \neg\forall x. \neg A$
- (57) $\vdash (\forall x. A) = \neg\exists x. \neg A$
- (58) $\vdash (\forall x. A \supset B) \supset ((\forall x. A) \supset (\forall x. B))$
- (59) $\vdash (B \vee \forall x. A) \supset (\forall x. A \vee B)$ with $x \notin \text{FV}(B)$
- (60) $\vdash (\exists x. A \supset B) \supset ((\forall x. A) \supset B)$ with $x \notin \text{FV}(B)$
- (61) $\vdash \exists x. B \supset A = B \supset \exists x. A$ with $x \notin \text{FV}(B)$
- (62) $\vdash A[t/x] = \exists x. x = t \wedge A$ with $x \notin \text{FV}(t)$
- (63) $\vdash A[t/x] = \forall x. x = t \supset A$ with $x \notin \text{FV}(t)$
- (64) $\vdash \forall x. \forall y. P = \forall y. \forall x. P$
- (65) $\vdash \exists x. \exists y. P = \exists y. \exists x. P$
- (66) $\vdash \forall x. A \vee B = (\forall x. A) \vee B$ with $x \notin \text{FV}(B)$
- (67) $\vdash (\forall x. A \supset B) \supset ((\exists x. A) \supset \exists x. B)$
- (68) $\vdash (\exists x. x = f(x) \wedge P[f(x)/x]) \supset \exists x. P$

1. SOLUTIONS

(1) $\vdash A \supset (B \supset A)$

$$\frac{\frac{[A]^1}{B \supset A} \supset I}{A \supset (B \supset A)} \supset I^1$$

(2) $\vdash (A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

$$\frac{[A \vee B]^1 \quad \frac{[A]^2 \quad [A \supset C]^3}{C} \supset E \quad \frac{[B]^2 \quad [B \supset C]^4}{C} \supset E}{C} \vee E^2$$

$$\frac{C}{A \vee B \supset C} \supset I^1$$

$$\frac{(B \supset C) \supset (A \vee B \supset C)}{(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))} \supset I^3$$

(3) $\vdash A = \neg\neg A$

$$\frac{\frac{A \vee \neg A}{[A]^1} \text{lem} \quad \frac{[\neg A]^1 \quad [\neg\neg A]^2}{\perp} \neg E}{\frac{A}{\neg\neg A \supset A} \supset I^2} \vee E^1$$

$$\frac{\frac{[\neg A]^1 \quad [A]^2}{\perp} \neg E}{\frac{\neg\neg A}{A \supset \neg\neg A} \supset I^2} \supset I^1$$

(4) $\vdash \neg A = \neg\neg\neg A$

$$\frac{[\neg\neg\neg A]^1 \quad \frac{\frac{[\neg A]^2 \quad [A]^3}{\perp} \neg E}{\neg\neg A} \supset I^2}{\neg A} \neg E$$

$$\frac{\perp}{\neg A} \neg I^3$$

$$\frac{\neg A}{\neg\neg\neg A \supset \neg A} \supset I^1$$

$$\frac{[\neg\neg A]^1 \quad [\neg A]^2}{\perp} \neg E$$

$$\frac{\perp}{\neg\neg\neg A} \supset I^1$$

$$\frac{\neg A}{\neg A \supset \neg\neg\neg A} \supset I^2$$

(5) $\vdash \neg(A \vee B) = \neg A \wedge \neg B$

$$\begin{array}{c}
 \frac{\frac{[\neg(A \vee B)]^1}{\perp} \neg I^2}{\neg A} \neg I^2 \quad \frac{\frac{[B]^3}{A \vee B} \vee I_2}{[\neg(A \vee B)]^1 \neg E} \neg E \\
 \frac{\neg A \quad \neg B}{\neg A \wedge \neg B} \wedge I \\
 \frac{\neg(A \vee B) \supset \neg A \wedge \neg B}{\neg(A \vee B) \supset \neg A \wedge \neg B} \supset I^1 \\
 \frac{\frac{[A]^2}{\neg A} \neg E \quad \frac{[B]^2}{\neg B} \neg E}{\perp} \vee E^2 \\
 \frac{\perp}{\neg(A \vee B)} \neg I^1 \\
 \frac{\neg(A \vee B) \supset \neg(A \vee B)}{\neg A \wedge \neg B \supset \neg(A \vee B)} \supset I^3
 \end{array}$$

(6) $\vdash (A \supset B) \vee (B \supset A)$

$$\begin{array}{c}
 \frac{[A]^2 \quad [\neg A]^1}{\perp} \neg E \\
 \frac{\perp}{B} \perp E \\
 \frac{B \supset B}{A \supset B} \supset I^2 \\
 \frac{A \vee \neg A}{(A \supset B) \vee (B \supset A)} \vee I_2 \quad \frac{A \supset B}{(A \supset B) \vee (B \supset A)} \vee I_1 \\
 \frac{(A \supset B) \vee (B \supset A)}{(A \supset B) \vee (B \supset A)} \vee E^1
 \end{array}$$

(7) $\vdash \neg A \supset (A \supset B)$

$$\begin{array}{c}
 \frac{[\neg A]^1 \quad [A]^2}{\perp} \neg E \\
 \frac{\perp}{B} \perp E \\
 \frac{B}{A \supset B} \supset I^2 \\
 \frac{A \supset B}{\neg A \supset (A \supset B)} \supset I^1
 \end{array}$$

(8) $\vdash ((A \supset B) \supset A) \supset A$

$$\begin{array}{c}
 \frac{[\neg A]^1 \quad [A]^3}{\perp} \neg E \\
 \frac{\perp}{B} \perp E \\
 \frac{B}{A \supset B} \supset I^3 \\
 \frac{A \vee \neg A}{[A]^1 \quad A} \vee E^1 \quad \frac{[(A \supset B) \supset A]^2}{A \supset B} \supset E \\
 \frac{A}{((A \supset B) \supset A) \supset A} \supset I^2
 \end{array}$$

(9) $\vdash \neg\neg(A \supset B) = \neg\neg A \supset \neg\neg B$

$$\begin{array}{c}
\frac{[A]^1 \quad [A \supset B]^2}{B} \supset E \quad \frac{[\neg B]^3}{\quad} \neg E \\
\hline
\frac{\perp}{\neg(A \supset B)} \neg I^2 \quad \frac{[\neg\neg(A \supset B)]^4}{\quad} \neg E \\
\hline
\frac{\perp}{\neg A} \neg I^1 \quad \frac{[\neg\neg A]^5}{\quad} \neg E \\
\hline
\frac{\perp}{\neg\neg B} \neg I^3 \quad \frac{\neg\neg A \supset \neg\neg B}{\quad} \supset I^5 \\
\hline
\frac{\neg\neg(A \supset B) \supset \neg\neg A \supset \neg\neg B}{\quad} \supset I^4 \\
\hline
\frac{[A]^3 \quad [\neg A]^4}{\neg\neg A} \neg E \quad \frac{[\neg(A \supset B)]^1 \quad A \supset B}{\neg B} \supset E \\
\frac{[\neg\neg A \supset \neg\neg B]^2}{\neg\neg B} \supset E \quad \frac{\perp}{\neg B} \neg I^5 \\
\hline
\frac{\perp}{B} \perp E \quad \frac{A \supset B}{\quad} \supset I^3 \\
\hline
\frac{[\neg(A \supset B)]^1}{\neg\neg(A \supset B)} \neg I^1 \quad \frac{A \supset B}{\quad} \supset I^2 \\
\hline
\frac{\neg\neg(A \supset B) \supset \neg\neg(A \supset B)}{\quad} \supset I^2
\end{array}$$

(10) $\vdash A \supset B = \neg B \supset \neg A$

$$\begin{array}{c}
\frac{[A \supset B]^1 \quad [A]^2}{B} \supset E \quad \frac{[\neg B]^3}{\quad} \neg E \quad \frac{[\neg B \supset \neg A]^2 \quad [\neg B]^1}{\neg A} \supset E \quad \frac{[A]^3}{\quad} \neg E \\
\hline
\frac{\perp}{\neg A} \neg I^2 \quad \frac{B \vee \neg B}{\quad} \text{lem} \quad \frac{[B]^1}{\quad} \supset I^3 \quad \frac{\perp}{B} \perp E \\
\hline
\frac{\neg B \supset \neg A}{\quad} \supset I^3 \quad \frac{A \supset B}{\quad} \supset I^3 \quad \frac{B}{\neg B \supset \neg A} \vee E^1 \\
\hline
\frac{(A \supset B) \supset (\neg B \supset \neg A)}{\quad} \supset I^1 \quad \frac{(\neg B \supset \neg A) \supset (A \supset B)}{\quad} \supset I^2
\end{array}$$

(11) $\vdash A \vee B = \neg A \supset B$

$$\begin{array}{c}
\frac{[A]^2 \quad [\neg A]^3}{\quad} \neg E \quad \frac{\perp}{B} \perp E \quad \frac{[B]^2}{\quad} \vee E^2 \quad \frac{[\neg A]^1 \quad [\neg A \supset B]^2}{B} \supset E \\
\hline
\frac{A \vee B}{\quad} \supset I^3 \quad \frac{A \vee \neg A}{\quad} \text{lem} \quad \frac{[A]^1}{A \vee B} \vee I_1 \quad \frac{B}{A \vee B} \vee I_2 \\
\hline
\frac{\neg A \supset B}{\quad} \supset I^1 \quad \frac{A \vee B}{\quad} \supset I^2 \quad \frac{(\neg A \supset B) \supset A \vee B}{\quad} \supset I^2
\end{array}$$

$$(12) \vdash A \supset B = \neg(A \wedge \neg B)$$

$$\begin{array}{c}
 \frac{\frac{[A \supset B]^1}{B} \quad \frac{\frac{[A \wedge \neg B]^2}{A} \wedge E_1}{\supset E} \quad \frac{\frac{[A \wedge \neg B]^2}{\neg B} \wedge E_2}{\neg E}}{\frac{\perp}{\neg(A \wedge \neg B)} \neg I^2} \neg E \\
 \frac{\neg(A \wedge \neg B)}{(A \supset B) \supset \neg(A \wedge \neg B)} \supset I^1 \\
 \frac{\frac{[A]^2 \quad [\neg B]^1}{A \wedge \neg B} \wedge I \quad [\neg(A \wedge \neg B)]^3}{\perp} \neg E \\
 \frac{\frac{B \vee \neg B}{[B]^1} \text{lem} \quad \frac{\perp}{B} \perp E}{B} \vee E^1 \\
 \frac{B}{A \supset B} \supset I^2 \\
 \frac{A \supset B}{\neg(A \wedge \neg B) \supset (A \supset B)} \supset I^3
 \end{array}$$

$$(13) \vdash A \wedge B \supset \neg(A \supset \neg B)$$

$$\begin{array}{c}
 \frac{\frac{[A \supset \neg B]^1}{\neg B} \quad \frac{\frac{[A \wedge B]^2}{A} \wedge E_1}{\supset E} \quad \frac{[A \wedge B]^2}{B} \wedge E_2}{\perp} \neg E \\
 \frac{\perp}{\neg(A \supset \neg B)} \neg I^1 \\
 \frac{\neg(A \supset \neg B)}{A \wedge B \supset \neg(A \supset \neg B)} \supset I^2
 \end{array}$$

$$(14) \vdash (A \vee B) \vee C = A \vee (B \vee C)$$

$$\begin{array}{c}
 \frac{[(A \vee B) \vee C]^1 \quad \frac{\frac{[A]^3}{A \vee (B \vee C)} \vee I_1 \quad \frac{\frac{[B]^3}{B \vee C} \vee I_1}{A \vee (B \vee C)} \vee I_2 \quad \frac{[C]^2}{B \vee C} \vee I_2}{A \vee (B \vee C)} \vee E^3}{A \vee (B \vee C)} \vee E^2 \\
 \frac{A \vee (B \vee C)}{(A \vee B) \vee C \supset A \vee (B \vee C)} \supset I^1 \\
 \frac{[A \vee (B \vee C)]^1 \quad \frac{\frac{[A]^2}{A \vee B} \vee I_1}{(A \vee B) \vee C} \vee I_1 \quad \frac{\frac{[B]^3}{A \vee B} \vee I_2 \quad \frac{[C]^3}{(A \vee B) \vee C} \vee I_2}{(A \vee B) \vee C} \vee I_2}{(A \vee B) \vee C} \vee E^3 \\
 \frac{(A \vee B) \vee C}{A \vee (B \vee C) \supset (A \vee B) \vee C} \supset I^1
 \end{array}$$

(15) $\vdash \neg\neg(A \wedge B) = \neg\neg A \wedge \neg\neg B$

$$\begin{array}{c}
 \frac{\frac{\frac{[\neg A]^2}{\perp} \neg I^3}{[\neg\neg(A \wedge B)]^1} \neg E}{\frac{\perp}{\neg\neg A} \neg I^2} \neg E \quad \frac{\frac{\frac{[\neg B]^4}{\perp} \neg I^5}{[\neg\neg(A \wedge B)]^1} \neg E}{\frac{\perp}{\neg\neg B} \neg I^4} \neg E \\
 \frac{\frac{\frac{[A \wedge B]^3}{\perp} \neg E}{\neg\neg A \wedge \neg\neg B} \wedge I}{\neg\neg(A \wedge B) \supset \neg\neg A \wedge \neg\neg B} \supset I^1 \\
 \frac{\frac{\frac{[A]^1 [B]^2}{A \wedge B} \wedge I}{\frac{\perp}{\neg A} \neg I^1} \neg E \quad \frac{\frac{[\neg\neg A \wedge \neg\neg B]^4}{\neg\neg A} \wedge E_1}{\frac{\perp}{\neg B} \neg I^2} \neg E \\
 \frac{\frac{\frac{[\neg\neg A \wedge \neg\neg B]^4}{\neg\neg B} \wedge E_2}{\frac{\perp}{\neg\neg(A \wedge B)} \neg I^3} \neg E \\
 \frac{\neg\neg A \wedge \neg\neg B \supset \neg\neg(A \wedge B)}{\neg\neg(A \wedge B) = \neg\neg A \wedge \neg\neg B} =I
 \end{array}$$

(16) $\vdash (A \wedge B) \wedge C = A \wedge (B \wedge C)$

$$\begin{array}{c}
 \frac{\frac{[(A \wedge B) \wedge C]^1}{A \wedge B} \wedge E_1}{A} \wedge E_1 \quad \frac{\frac{[(A \wedge B) \wedge C]^1}{B} \wedge E_2}{B \wedge C} \wedge I \\
 \frac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C \supset A \wedge (B \wedge C)} \supset I^1 \\
 \frac{\frac{[A \wedge (B \wedge C)]^1}{A} \wedge E_1}{A \wedge B} \wedge I \quad \frac{\frac{[A \wedge (B \wedge C)]^1}{B \wedge C} \wedge E_2}{C} \wedge E_2 \\
 \frac{(A \wedge B) \wedge C}{A \wedge (B \wedge C) \supset (A \wedge B) \wedge C} \supset I^1
 \end{array}$$

(17) $\vdash A \vee B = B \vee A$ We observe that the property is auto-dual

$$\begin{array}{c}
 \frac{[A]^2}{B \vee A} \vee I_2 \quad \frac{[B]^2}{B \vee A} \vee I_1 \\
 \frac{[A \vee B]^1}{B \vee A} \vee E^2 \\
 \frac{B \vee A}{A \vee B \supset B \vee A} \supset I^1
 \end{array}$$

(18) $\vdash A \wedge B = B \wedge A$ We observe that the property is auto-dual

$$\frac{\frac{\frac{[A \wedge B]^1}{B} \wedge E_2 \quad \frac{[A \wedge B]^1}{A} \wedge E_1}{B \wedge A} \wedge I}{A \wedge B \supset B \wedge A} \supset I^1$$

(19) $\vdash A \vee A = A$

$$\frac{\frac{\frac{[A \vee A]^1 \quad [A]^2 \quad [A]^2}{A} \vee E^2}{A \vee A \supset A} \supset I^1 \quad \frac{\frac{[A]^1}{A \vee A} \vee I_1}{A \supset A \vee A} \supset I^1 \quad \frac{\frac{[A]^1}{A \vee A} \vee I_2}{A \supset A \vee A} \supset I^1$$

(20) $\vdash A \wedge A = A$

$$\frac{\frac{\frac{[A \wedge A]^1}{A} \wedge E_1}{A \wedge A \supset A} \supset I^1 \quad \frac{\frac{\frac{[A \wedge A]^1}{A} \wedge E_2}{A \wedge A \supset A} \supset I^1 \quad \frac{\frac{[A]^1 \quad [A]^1}{A \wedge A} \wedge I}{A \supset A \wedge A} \supset I^1$$

(21) $\vdash A \vee (A \wedge B) = A$

$$\frac{\frac{\frac{[A \vee (A \wedge B)]^1 \quad [A]^2}{A} \vee E^2 \quad \frac{\frac{[A \wedge B]^2}{A} \wedge E_1}{A \vee (A \wedge B) \supset A} \supset I^1 \quad \frac{\frac{[A]^1}{A \vee (A \wedge B)} \vee I_1}{A \supset A \vee (A \wedge B)} \supset I^1$$

(22) $\vdash A \wedge (A \vee B) = A$

$$\frac{\frac{\frac{[A \wedge (A \vee B)]^1}{A} \wedge E_1}{A \wedge (A \vee B) \supset A} \supset I^1 \quad \frac{\frac{[A]^1 \quad \frac{[A]^1}{A \vee B} \vee I_1}{A \wedge (A \vee B)} \wedge I}{A \supset A \wedge (A \vee B)} \supset I^1$$

$$(23) \vdash A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$\begin{array}{c}
 \frac{[A \wedge (B \vee C)]^1}{B \vee C} \wedge E_2 \quad \frac{\frac{[A \wedge (B \vee C)]^1}{A} \wedge E_1 \quad \frac{[B]^2}{A \wedge B} \wedge I}{(A \wedge B) \vee (A \wedge C)} \vee I_1 \quad \frac{\frac{[A \wedge (B \vee C)]^1}{A} \wedge E_1 \quad \frac{[C]^2}{A \wedge C} \wedge I}{(A \wedge B) \vee (A \wedge C)} \vee I_2 \\
 \hline
 \frac{(A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \supset (A \wedge B) \vee (A \wedge C)} \supset I^1 \\
 \frac{[(A \wedge B) \vee (A \wedge C)]^1}{A \wedge (B \vee C)} \wedge E_1 \quad \frac{\frac{[A \wedge B]^2}{A} \wedge E_1 \quad \frac{[A \wedge B]^2}{B \vee C} \vee I_1}{A \wedge (B \vee C)} \wedge I \quad \frac{\frac{[A \wedge C]^2}{A} \wedge E_1 \quad \frac{[A \wedge C]^2}{B \vee C} \vee I_2}{A \wedge (B \vee C)} \wedge I \\
 \hline
 \frac{A \wedge (B \vee C)}{(A \wedge B) \vee (A \wedge C) \supset A \wedge (B \vee C)} \supset I^1
 \end{array}$$

$$(24) \vdash A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

$$\begin{array}{c}
 \frac{[A]^2}{A \vee B} \vee I_1 \quad \frac{[A]^2}{A \vee C} \vee I_1 \quad \frac{\frac{[B \wedge C]^2}{B} \wedge E_1 \quad \frac{[B \wedge C]^2}{C} \wedge E_2}{A \vee B} \vee I_2 \quad \frac{\frac{[B \wedge C]^2}{B} \wedge E_1 \quad \frac{[B \wedge C]^2}{C} \wedge E_2}{A \vee C} \vee I_2 \\
 \hline
 \frac{[A \vee (B \wedge C)]^1}{(A \vee B) \wedge (A \vee C)} \wedge I^1 \\
 \frac{(A \vee B) \wedge (A \vee C)}{A \vee (B \wedge C) \supset (A \vee B) \wedge (A \vee C)} \supset I^1 \\
 \frac{[(A \vee B) \wedge (A \vee C)]^1}{A \vee B} \wedge E_1 \quad \frac{[A]^2}{A \vee (B \wedge C)} \vee I_1 \quad \frac{\frac{[(A \vee B) \wedge (A \vee C)]^1}{A \vee C} \wedge E_2 \quad \frac{[A]^3}{A \vee (B \wedge C)} \vee I_1}{A \vee (B \wedge C)} \vee I_2 \\
 \hline
 \frac{A \vee (B \wedge C)}{(A \vee B) \wedge (A \vee C) \supset A \vee (B \wedge C)} \supset I^1
 \end{array}$$

(25) $\vdash \neg(A \vee B) = \neg A \wedge \neg B$

$$\begin{array}{c}
 \frac{\frac{[A]^1}{A \vee B} \vee I_1 \quad \frac{[B]^3}{A \vee B} \vee I_2 \quad \frac{[\neg(A \vee B)]^2}{\perp} \neg E}{\frac{A \vee \neg A}{\neg A \wedge \neg B} \text{lem} \quad \frac{\perp}{\neg A \wedge \neg B} \neg E} \neg E \quad \frac{\frac{B \vee \neg B}{\neg A \wedge \neg B} \text{lem} \quad \frac{\perp}{\neg A \wedge \neg B} \neg E \quad \frac{[\neg A]^1 \quad [\neg B]^3}{\neg A \wedge \neg B} \wedge I}{\neg A \wedge \neg B} \vee E^3 \\
 \frac{\neg A \wedge \neg B}{\neg(A \vee B) \supset \neg A \wedge \neg B} \supset I^2 \\
 \frac{[A \vee B]^1 \quad \frac{[A]^2 \quad \frac{[\neg A \wedge \neg B]^3}{\neg A} \wedge E_1}{\perp} \neg E \quad \frac{[B]^2 \quad \frac{[\neg A \wedge \neg B]^3}{\neg B} \wedge E_2}{\perp} \neg E}{\perp} \vee E^2 \\
 \frac{\perp}{\neg(A \vee B)} \neg I^1 \\
 \frac{\neg(A \vee B) \supset \neg A \wedge \neg B}{\neg A \wedge \neg B \supset \neg(A \vee B)} \supset I^3
 \end{array}$$

(26) $\vdash \neg(A \wedge B) = \neg A \vee \neg B$

$$\begin{array}{c}
 \frac{[A]^1 \quad [B]^2}{A \wedge B} \wedge I \quad \frac{[\neg(A \wedge B)]^3}{\perp} \neg E}{\frac{B \vee \neg B}{\neg A \vee \neg B} \text{lem} \quad \frac{\perp}{\neg A \vee \neg B} \neg E} \neg E \quad \frac{[\neg B]^2}{\neg A \vee \neg B} \vee I_2 \quad \frac{[\neg A]^1}{\neg A \vee \neg B} \vee I_1}{\neg A \vee \neg B} \vee E^1 \\
 \frac{\neg A \vee \neg B}{\neg(A \wedge B) \supset \neg A \vee \neg B} \supset I^3 \\
 \frac{[A \wedge B]^2}{A} \wedge E_1 \quad \frac{[\neg A]^3}{\perp} \neg E \quad \frac{[A \wedge B]^2}{B} \wedge E_2 \quad \frac{[\neg B]^3}{\perp} \neg E}{\perp} \vee E^3 \\
 \frac{\perp}{\neg(A \wedge B)} \neg I^2 \\
 \frac{\neg(A \wedge B) \supset \neg A \vee \neg B}{\neg A \vee \neg B \supset \neg(A \wedge B)} \supset I^1
 \end{array}$$

$$(27) \vdash A \wedge B = \neg(\neg A \vee \neg B)$$

$$\begin{array}{c}
 \frac{\frac{\frac{[A \wedge B]^3}{A} \wedge E_1 \quad \frac{[A \wedge B]^3}{B} \wedge E_2}{\frac{[\neg A]^2}{\perp} \neg E \quad \frac{[\neg B]^2}{\perp} \neg E} \vee E^2}{\frac{\perp}{\neg(\neg A \vee \neg B)} \neg I^1} \supset I^3 \\
 \frac{A \wedge B \supset \neg(\neg A \vee \neg B)}{\vdash} \\
 \frac{\frac{\frac{[A]^1}{A \wedge B} \wedge I \quad \frac{[B]^2}{A \wedge B} \wedge I}{\frac{[A \wedge B]^3}{\perp} \neg E} \neg E \quad \frac{\frac{[\neg A]^1}{\neg A \vee \neg B} \vee I_1 \quad \frac{[\neg(\neg A \vee \neg B)]^3}{\perp} \neg E}{\frac{[\neg A \vee \neg B]}{A \wedge B} \vee E^1} \neg E \\
 \frac{\frac{A \vee \neg A}{\vdash} \text{lem} \quad \frac{\frac{B \vee \neg B}{\vdash} \text{lem} \quad \frac{A \wedge B}{\vdash} \text{lem} \quad \frac{A \wedge B}{\vdash} \text{lem} \quad \frac{A \wedge B}{\vdash} \text{lem}}{\frac{A \wedge B}{\neg(\neg A \vee \neg B) \supset A \wedge B} \supset I^3} \supset I^3
 \end{array}$$

$$(28) \vdash A \vee B = \neg(\neg A \wedge \neg B)$$

$$\begin{array}{c}
 \frac{\frac{\frac{[\neg A \wedge \neg B]^2}{\neg A} \wedge E_1 \quad [A]^3}{\perp} \neg E \quad \frac{\frac{[\neg A \wedge \neg B]^2}{\neg B} \wedge E_2 \quad [B]^3}{\perp} \neg E}{\frac{[A \vee B]^1}{\perp} \neg E} \vee E^3 \\
 \frac{\perp}{\neg(\neg A \wedge \neg B)} \neg I^2 \\
 \frac{A \vee B \supset \neg(\neg A \wedge \neg B)}{\vdash} \supset I^1 \\
 \frac{\frac{[\neg A]^1}{\neg A \wedge \neg B} \wedge I \quad \frac{[\neg B]^2}{\neg(\neg A \wedge \neg B)} \neg E}{\frac{[\neg A \wedge \neg B]}{A \vee B} \vee E^1} \neg E \\
 \frac{\frac{A \vee \neg A}{\vdash} \text{lem} \quad \frac{[A]^1}{A \vee B} \vee I_1 \quad \frac{\frac{B \vee \neg B}{\vdash} \text{lem} \quad \frac{[B]^2}{A \vee B} \vee I_2}{\frac{A \vee B}{\vdash} \text{lem}} \vee E^1 \\
 \frac{A \vee B}{\neg(\neg A \wedge \neg B) \supset A \vee B} \supset I^3
 \end{array}$$

$$(29) \vdash A \supset B = \neg A \vee B$$

$$\begin{array}{c}
 \frac{\frac{[A \supset B]^1}{B} \supset E \quad [A]^2}{\frac{[\neg A]^2}{\neg A \vee B} \vee I_1} \vee E^2 \\
 \frac{\frac{A \vee \neg A}{\vdash} \text{lem} \quad \frac{B}{\neg A \vee B} \vee I_2}{\frac{\neg A \vee B}{(A \supset B) \supset \neg A \vee B} \supset I^1} \supset I^1 \\
 \frac{\frac{[\neg A]^2}{\neg A \vee B} \neg E \quad [A]^3}{\frac{[\neg A \vee B]^1}{B} \supset E} \supset E \\
 \frac{\frac{[\neg A \vee B]^1}{B} \supset E \quad [B]^2}{\frac{A \supset B}{\neg A \vee B \supset (A \supset B)} \supset I^1} \supset I^1
 \end{array}$$

(30) $\vdash (A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

$$\frac{\frac{\frac{[A \supset (B \supset C)]^1 \quad [A]^2}{B \supset C} \supset E \quad \frac{[A \supset B]^3 \quad [A]^2}{B} \supset E}{C} \supset E}{\frac{A \supset C}{(A \supset (B \supset C)) \supset (A \supset C)} \supset I^1} \supset I^3$$

(31) $\vdash A \supset (B \supset C) = B \supset (A \supset C)$ We observe that the property is auto-dual

$$\frac{\frac{\frac{[A \supset (B \supset C)]^1 \quad [A]^2}{B \supset C} \supset E \quad [B]^3}{C} \supset E}{\frac{A \supset C}{B \supset (A \supset C)} \supset I^3} \supset I^1$$

(32) $\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$

$$\frac{\frac{[B \supset C]^1 \quad \frac{[A \supset B]^2 \quad [A]^3}{B} \supset E}{C} \supset E}{\frac{A \supset C}{(B \supset C) \supset (A \supset C)} \supset I^1} \supset I^2$$

(33) $\vdash A \supset (B \supset C) = A \wedge B \supset C$

$$\frac{\frac{\frac{[A \supset (B \supset C)]^1 \quad \frac{[A \wedge B]^2}{A} \wedge E_1}{B \supset C} \supset E \quad \frac{[A \wedge B]^2}{B} \wedge E_2}{C} \supset E}{\frac{A \wedge B \supset C}{(A \supset (B \supset C)) \supset (A \wedge B \supset C)} \supset I^1} \supset I^1$$

$$\frac{\frac{[A \wedge B \supset C]^1 \quad \frac{[A]^2 \quad [B]^3}{A \wedge B} \wedge I}{C} \supset E}{\frac{B \supset C}{A \supset (B \supset C)} \supset I^3} \supset I^2$$

$$\frac{\frac{\frac{[A \wedge \neg A]^1}{\neg A} \wedge E_2}{A} \wedge E_1}{\perp} \neg E$$

$$\frac{}{\neg(A \wedge \neg A)} \neg I^1$$
[illegible]
$$\frac{\frac{\frac{[A \wedge B \supset C]^1}{C} \supset I^2}{A \supset C} \supset I^2 \quad \frac{\frac{\frac{[A]^2 \quad [B]^3}{A \wedge B} \wedge I \quad \frac{\frac{[\neg B]^3 \quad [B]^4}{\perp} \neg E}{C} \perp E}{B \supset C} \supset I^4}{(A \supset C) \vee (B \supset C)} \vee I_2}{(A \supset C) \vee (B \supset C)} \vee E^3 \quad \frac{B \vee \neg B}{(A \supset C) \vee (B \supset C)} \text{lem}}{(A \supset C) \vee (B \supset C)} \supset I^1$$
$$\frac{\frac{\frac{[A \supset \neg A]^1 \quad [A]^2}{\neg A} \supset E}{\perp} \neg I^2}{(A \supset \neg A) \supset \neg A} \supset I^1$$

$$(38) \vdash (A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)$$

$$\frac{\frac{A \vee \neg A}{\text{lem}} \quad \frac{\frac{[A \supset B \vee C]^1 \quad [A]^2}{B \vee C} \supset E \quad \frac{\frac{[B]^3}{A \supset B} \supset I \quad \frac{[C]^3}{A \supset C} \supset I}{(A \supset B) \vee (A \supset C)} \vee I_1 \quad \frac{\frac{[C]^3}{A \supset C} \supset I \quad \frac{[A]^2}{(A \supset B) \vee (A \supset C)} \vee I_2}{(A \supset B) \vee (A \supset C)} \vee E^3 \quad \frac{\frac{[\neg A]^2 \quad [A]^4}{\perp} \neg E \quad \frac{\perp}{B} \perp E}{A \supset B} \supset I^4}{(A \supset B) \vee (A \supset C)} \vee I_1 \quad \frac{(A \supset B) \vee (A \supset C)}{(A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)} \supset I^1$$

$$(39) \vdash (A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C)$$

$$\frac{[A \vee B]^1 \quad \frac{\frac{[(A \supset C) \wedge (B \supset C)]^2}{A \supset C} \wedge E_1 \quad [A]^3}{C} \supset E \quad \frac{\frac{[(A \supset C) \wedge (B \supset C)]^2}{B \supset C} \wedge E_2 \quad [B]^3}{C} \supset E}{C} \vee E^3 \quad \frac{C}{A \vee B \supset C} \supset I^1 \quad \frac{A \vee B \supset C}{(A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C)} \supset I^2$$

$$(40) \vdash (A \supset B) \wedge (A \supset C) \supset (A \supset B \wedge C)$$

$$\frac{\frac{[(A \supset B) \wedge (A \supset C)]^1}{A \supset B} \wedge E_1 \quad [A]^2}{B} \supset E \quad \frac{\frac{[(A \supset B) \wedge (A \supset C)]^1}{A \supset C} \wedge E_2 \quad [A]^2}{C} \supset E \quad \frac{B \quad C}{B \wedge C} \wedge I \quad \frac{B \wedge C}{A \supset B \wedge C} \supset I^2 \quad \frac{A \supset B \wedge C}{(A \supset B) \wedge (A \supset C) \supset (A \supset B \wedge C)} \supset I^1$$

$$(41) \vdash (A \supset B) \wedge (B \supset C) \supset (A \supset C)$$

$$\frac{\frac{[(A \supset B) \wedge (B \supset C)]^1}{B \supset C} \wedge E_2 \quad \frac{\frac{[(A \supset B) \wedge (B \supset C)]^1}{A \supset B} \wedge E_1 \quad [A]^2}{B} \supset E}{C} \supset E \quad \frac{C}{A \supset C} \supset I^2 \quad \frac{A \supset C}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)} \supset I^1$$

$$(42) \vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$$

$$\frac{\frac{[A \supset (B \supset C)]^1 \quad [A]^2}{B \supset C} \supset E \quad \frac{[A \supset B]^3 \quad [A]^2}{B} \supset E}{C} \supset E \quad \frac{C}{A \supset C} \supset I^2 \quad \frac{A \supset C}{(A \supset B) \supset (A \supset C)} \supset I^3 \quad \frac{(A \supset B) \supset (A \supset C)}{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))} \supset I^1$$

(43) $\vdash \forall x. B \supset A = B \supset \forall x. A$ with $x \notin \text{FV}(B)$

$$\frac{\frac{[B]^1 \quad \frac{[\forall x. B \supset A]^2}{B \supset A} \vee E}{A} \supset E \quad \frac{A}{\forall x. A} \vee I}{B \supset \forall x. A} \supset I^1 \quad \frac{(\forall x. B \supset A) \supset (B \supset \forall x. A)}{} \supset I^2 \quad \frac{[B \supset \forall x. A]^1 \quad [B]^2}{\forall x. A} \supset E \quad \frac{\forall x. A}{A} \vee E \quad \frac{A}{B \supset A} \supset I^2 \quad \frac{B \supset A}{\forall x. B \supset A} \vee I}{(B \supset \forall x. A) \supset (\forall x. B \supset A)} \supset I^1$$

(44) $\vdash \forall x. A \supset B = (\exists x. A) \supset B$ with $x \notin \text{FV}(B)$

$$\frac{[\exists x. A]^1 \quad \frac{[A]^2 \quad \frac{[\forall x. A \supset B]^3}{A \supset B} \vee E}{B} \supset E}{B} \exists E^2 \quad \frac{B}{(\exists x. A) \supset B} \supset I^1 \quad \frac{(\forall x. A \supset B) \supset ((\exists x. A) \supset B)}{} \supset I^3 \quad \frac{[A]^1}{\exists x. A} \exists I \quad \frac{[(\exists x. A) \supset B]^2}{B} \supset E \quad \frac{B}{A \supset B} \supset I^1 \quad \frac{A \supset B}{\forall x. A \supset B} \vee I}{((\exists x. A) \supset B) \supset (\forall x. A \supset B)} \supset I^2$$

(45) $\vdash \neg \neg \forall x. A \supset \forall x. \neg \neg A$

$$\frac{[\forall x. A]^1}{A} \vee E \quad \frac{A \quad [\neg A]^2}{\perp} \neg E \quad \frac{\perp}{\neg \forall x. A} \neg I^1 \quad \frac{[\neg \neg \forall x. A]^3}{\perp} \neg E \quad \frac{\perp}{\neg \neg A} \neg I^2 \quad \frac{\neg \neg A}{\forall x. \neg \neg A} \vee I \quad \frac{\neg \neg (\forall x. A) \supset \forall x. \neg \neg A}{\neg \neg \forall x. A \supset \forall x. \neg \neg A} \supset I^3$$

(46) $\vdash A \wedge \forall x. B = \forall x. A \wedge B$ with $x \notin \text{FV}(A)$

$$\frac{[A \wedge \forall x. B]^1}{A} \wedge E_1 \quad \frac{[A \wedge \forall x. B]^1}{\forall x. B} \wedge E_2 \quad \frac{A \quad B}{A \wedge B} \wedge I \quad \frac{A \wedge B}{\forall x. A \wedge B} \vee I \quad \frac{A \wedge (\forall x. B) \supset \forall x. A \wedge B}{\forall x. A \wedge B} \supset I^1 \quad \frac{[\forall x. A \wedge B]^1}{\forall x. A \wedge B} \vee E \quad \frac{\forall x. A \wedge B}{A \wedge B} \wedge E_2 \quad \frac{A \wedge B}{A} \wedge E_1 \quad \frac{A}{\forall x. B} \vee I \quad \frac{A \wedge \forall x. B}{(\forall x. A \wedge B) \supset A \wedge \forall x. B} \supset I^1$$

$$\begin{array}{c}
\frac{[A \wedge \exists x. B]^1}{A} \wedge E_1 \quad \frac{[B]^2}{[B]^2} \wedge I \\
\frac{[A \wedge \exists x. B]^1}{\exists x. B} \wedge E_2 \quad \frac{A \wedge B}{\exists x. A \wedge B} \exists I \\
\hline
\frac{\exists x. A \wedge B}{A \wedge (\exists x. B) \supset \exists x. A \wedge B} \supset I^1 \\
\\
\frac{[A \wedge B]^2}{B} \wedge E_2 \quad \frac{[A \wedge B]^2}{A} \wedge E_1 \\
\frac{B}{\exists x. B} \exists I \quad \frac{A}{A \wedge \exists x. B} \wedge I \\
\hline
\frac{[\exists x. A \wedge B]^1}{A \wedge \exists x. B} \exists E^2 \\
\hline
\frac{A \wedge \exists x. B}{(\exists x. A \wedge B) \supset A \wedge \exists x. B} \supset I^1
\end{array}$$
$$\begin{array}{c}
\frac{[B]^3}{\exists x. B} \exists E \\
\frac{[A]^3}{A \vee \exists x. B} \vee I_1 \quad \frac{[A \vee \exists x. B]^2}{A \vee \exists x. B} \vee I_2 \\
\frac{[\exists x. A \vee B]^1}{A \vee \exists x. B} \exists E^2 \\
\frac{A \vee \exists x. B}{(\exists x. A \vee B) \supset A \vee \exists x. B} \supset I^1 \\
\\
\frac{[B]^3}{A \vee B} \vee I_2 \\
\frac{[A]^2}{A \vee B} \vee I_1 \quad \frac{[\exists x. A \vee B]^2}{\exists x. A \vee B} \exists I \\
\frac{[A \vee \exists x. B]^1}{\exists x. A \vee B} \exists I \\
\frac{\exists x. A \vee B}{A \vee (\exists x. B) \supset \exists x. A \vee B} \supset I^1
\end{array}$$
[illegible]

(50) $\vdash \forall x. A \wedge B = (\forall x. A) \wedge (\forall x. B)$

$$\begin{array}{c}
 \frac{\frac{\frac{[\forall x. A \wedge B]^1}{A \wedge B} \forall E}{A} \wedge E_1}{\forall x. A} \forall I \quad \frac{\frac{\frac{[\forall x. A \wedge B]^1}{A \wedge B} \forall E}{B} \wedge E_2}{\forall x. B} \forall I \\
 \hline
 (\forall x. A) \wedge (\forall x. B) \wedge I \\
 \hline
 (\forall x. A \wedge B) \supset (\forall x. A) \wedge (\forall x. B) \supset I^1 \\
 \hline
 \frac{[(\forall x. A) \wedge (\forall x. B)]^1}{\forall x. A} \wedge E_1 \quad \frac{[(\forall x. A) \wedge (\forall x. B)]^1}{\forall x. B} \wedge E_2 \\
 \hline
 \frac{\frac{\forall x. A}{A} \forall E}{A \wedge B} \wedge I \quad \frac{\forall x. B}{B} \forall E \\
 \hline
 \frac{A \wedge B}{\forall x. A \wedge B} \forall I \\
 \hline
 (\forall x. A) \wedge (\forall x. B) \supset \forall x. A \wedge B \supset I^1
 \end{array}$$

(51) $\vdash (\exists x. A \wedge B) \supset (\exists x. A) \wedge (\exists x. B)$

$$\begin{array}{c}
 \frac{[A \wedge B]^2}{A} \wedge E_1 \quad \frac{[A \wedge B]^2}{B} \wedge E_2 \\
 \hline
 \frac{A}{\exists x. A} \exists I \quad \frac{B}{\exists x. B} \exists I \\
 \hline
 \frac{[\exists x. A \wedge B]^1}{(\exists x. A) \wedge (\exists x. B)} \wedge I \\
 \hline
 (\exists x. A) \wedge (\exists x. B) \exists E^2 \\
 \hline
 (\exists x. A \wedge B) \supset (\exists x. A) \wedge (\exists x. B) \supset I^1
 \end{array}$$

(52) $\vdash (\exists x. A \wedge B) \supset (\exists x. A) \vee (\exists x. B)$ Both these proofs are correct:

$$\begin{array}{c}
 \frac{[A \wedge B]^2}{A} \wedge E_1 \quad \frac{[A \wedge B]^2}{B} \wedge E_2 \\
 \hline
 \frac{A}{\exists x. A} \exists I \quad \frac{B}{\exists x. B} \exists I \\
 \hline
 \frac{[\exists x. A \wedge B]^1}{(\exists x. A) \vee (\exists x. B)} \vee I_1 \quad \frac{[\exists x. A \wedge B]^1}{(\exists x. A) \vee (\exists x. B)} \vee I_1 \\
 \hline
 (\exists x. A) \vee (\exists x. B) \exists E^2 \quad (\exists x. A) \vee (\exists x. B) \exists E^2 \\
 \hline
 (\exists x. A \wedge B) \supset (\exists x. A) \vee (\exists x. B) \supset I^1 \quad (\exists x. A \wedge B) \supset (\exists x. A) \vee (\exists x. B) \supset I^1
 \end{array}$$

$$(53) \vdash \exists x. A \vee B = (\exists x. A) \vee (\exists x. B)$$

$$\frac{\frac{\frac{[A]^3}{\exists x. A} \exists I}{[A \vee B]^2} \vee I_1 \quad \frac{\frac{[B]^3}{\exists x. B} \exists I}{(\exists x. A) \vee (\exists x. B)} \vee I_2}{\frac{[\exists x. A \vee B]^1}{(\exists x. A) \vee (\exists x. B)} \exists E^2} \vee E^3$$

$$\frac{(\exists x. A) \vee (\exists x. B)}{(\exists x. A \vee B) \supset (\exists x. A) \vee (\exists x. B)} \supset I^1$$

$$\frac{\frac{[A]^3}{A \vee B} \vee I_1 \quad \frac{[B]^4}{A \vee B} \vee I_2}{\frac{[\exists x. A]^2}{\exists x. A \vee B} \exists I \quad \frac{[\exists x. B]^2}{\exists x. A \vee B} \exists I} \exists E^3 \quad \exists E^4$$

$$\frac{[(\exists x. A) \vee (\exists x. B)]^1}{\exists x. A \vee B} \vee E^2$$

$$\frac{(\exists x. A) \vee (\exists x. B) \supset (\exists x. A \vee B)}{(\exists x. A) \vee (\exists x. B) \supset (\exists x. A \vee B)} \supset I^1$$

$$(54) \vdash (\forall x. A) \supset \exists x. A$$

$$\frac{\frac{[\forall x. A]^1}{A} \forall E}{\exists x. A} \exists I$$

$$\frac{(\forall x. A) \supset \exists x. A}{(\forall x. A) \supset \exists x. A} \supset I^1$$

$$(55) \vdash (\exists x. \forall y. A) \supset \forall y. \exists x. A$$

$$\frac{\frac{[\forall y. A]^2}{A} \forall E}{\exists x. A} \exists I$$

$$\frac{[\exists x. \forall y. A]^1}{\forall y. \exists x. A} \forall I$$

$$\frac{\forall y. \exists x. A}{(\exists x. \forall y. A) \supset \forall y. \exists x. A} \supset I^1$$

$$(56) \vdash (\exists x. A) = \neg \forall x. \neg A$$

$$\frac{\frac{[\exists x. A]^1}{\perp} \exists E^2 \quad \frac{[A]^2}{\neg A} \neg E}{\perp} \neg I^3$$

$$\frac{\perp}{\neg \forall x. \neg A} \neg I^3$$

$$\frac{\neg \forall x. \neg A}{(\exists x. A) \supset \neg \forall x. \neg A} \supset I^1$$

$$\frac{\frac{[\neg \exists x. A]^1}{\perp} \neg E \quad \frac{[A]^2}{\exists x. A} \exists I}{\perp} \neg I^2$$

$$\frac{\perp}{\forall x. \neg A} \forall I$$

$$\frac{\forall x. \neg A}{\neg(\forall x. \neg A)} \neg E$$

$$\frac{(\exists x. A) \vee \neg \exists x. A \text{ lem} \quad \frac{[\exists x. A]^1}{\exists x. A} \exists I}{\exists x. A} \vee E^1$$

$$\frac{\neg(\forall x. \neg A) \supset \exists x. A}{\neg(\forall x. \neg A) \supset \exists x. A} \supset I^3$$

(57) $\vdash (\forall x. A) = \neg \exists x. \neg A$

$$\begin{array}{c}
 \frac{\frac{\frac{[\exists x. \neg A]^1}{\perp} \neg E^2}{\neg \exists x. \neg A} \neg I^1}{(\forall x. A) \supset \neg \exists x. \neg A} \supset I^3 \qquad \frac{\frac{\frac{[\neg A]^2}{A} \neg E}{A \vee \neg A} \text{lem} \quad \frac{\frac{[\neg(\exists x. \neg A)]^2}{\exists x. \neg A} \exists I \quad \frac{[\neg A]^1}{\exists x. \neg A} \neg E}{\frac{\frac{A}{\forall x. A} \forall I}{\neg(\exists x. \neg A) \supset \forall x. A} \supset I^2} \frac{\perp}{A} \perp E}{\forall x. A} \forall E^1
 \end{array}$$

(58) $\vdash (\forall x. A \supset B) \supset ((\forall x. A) \supset (\forall x. B))$

$$\begin{array}{c}
 \frac{\frac{[\forall x. A \supset B]^1}{A \supset B} \forall E \quad \frac{[\forall x. A]^2}{A} \forall E}{B} \supset E \\
 \frac{B}{\forall x. B} \forall I \\
 \frac{\forall x. B}{(\forall x. A) \supset \forall x. B} \supset I^2 \\
 \frac{(\forall x. A) \supset \forall x. B}{(\forall x. A \supset B) \supset ((\forall x. A) \supset \forall x. B)} \supset I^1
 \end{array}$$

(59) $\vdash (B \vee \forall x. A) \supset (\forall x. A \vee B)$ with $x \notin \text{FV}(B)$

$$\begin{array}{c}
 \frac{\frac{[B]^2}{A \vee B} \vee I_1 \quad \frac{\frac{[\forall x. A]^2}{A} \forall E}{A \vee B} \vee I_2}{\forall x. A \vee B} \forall I \\
 \frac{[B \vee \forall x. A]^1}{\forall x. A \vee B} \forall E^2 \\
 \frac{\forall x. A \vee B}{(B \vee \forall x. A) \supset (\forall x. A \vee B)} \supset I^1
 \end{array}$$

(60) $\vdash (\exists x. A \supset B) \supset ((\forall x. A) \supset B)$ with $x \notin \text{FV}(B)$

$$\begin{array}{c}
 \frac{\frac{[A \supset B]^2}{A} \supset E \quad \frac{[\forall x. A]^3}{A} \forall E}{B} \supset E \\
 \frac{B}{(\forall x. A) \supset B} \supset I^3 \\
 \frac{[\exists x. A \supset B]^1}{(\forall x. A) \supset B} \exists E^2 \\
 \frac{(\forall x. A) \supset B}{(\exists x. A \supset B) \supset ((\forall x. A) \supset B)} \supset I^1
 \end{array}$$

$$\begin{array}{c}
\frac{[B \supset A]^2 \quad [B]^3}{A} \supset E \\
\frac{[\exists x. B \supset A]^1 \quad \frac{A}{\exists x. A} \exists I}{\exists x. A} \exists E^2 \\
\frac{\exists x. A}{B \supset \exists x. A} \supset I^3 \\
\frac{B \supset \exists x. A}{(\exists x. B \supset A) \supset (B \supset \exists x. A)} \supset I^1 \\
\\
\frac{[A]^3}{B \supset A} \supset I \quad \frac{[\neg B]^2 \quad [B]^4}{\perp} \neg E \\
\frac{[B \supset \exists x. A]^1 \quad [B]^2}{\exists x. A} \supset E \quad \frac{B \supset A}{\exists x. B \supset A} \exists I \quad \frac{\perp}{A} \perp E \\
\frac{\exists x. A}{\exists x. B \supset A} \exists E^3 \quad \frac{B \supset A}{\exists x. B \supset A} \supset I^4 \\
\frac{B \vee \neg B}{\exists x. B \supset A} \text{lem} \quad \frac{\exists x. B \supset A}{\exists x. B \supset A} \exists E^3 \\
\frac{\exists x. B \supset A}{(B \supset \exists x. A) \supset (\exists x. B \supset A)} \supset I^1
\end{array}$$
$$\begin{array}{c}
\frac{}{\forall x. x = x} \text{refl} \\
\frac{}{\forall E} \\
\frac{t = t \quad [A[t/x]]^1}{t = t \wedge A[t/x]} \wedge I \\
\frac{}{\exists I} \\
\frac{\exists x. x = t \wedge A}{A[t/x] \supset \exists x. x = t \wedge A} \supset I^1 \\
\frac{[x = t \wedge A]^2}{x = t} \wedge E_1 \quad \frac{[x = t \wedge A]^2}{A} \wedge E_2 \\
\frac{[\exists x. x = t \wedge A]^1 \quad A[t/x]}{A[t/x]} \text{subst} \\
\frac{}{\exists E^2} \\
\frac{A[t/x]}{(\exists x. x = t \wedge A) \supset A[t/x]} \supset I^1
\end{array}$$

$$\begin{array}{c}
\frac{}{\forall x. \forall y. x = y \supset y = x} \text{sym} \\
\frac{}{\forall y. x = y \supset y = x} \forall E \\
\frac{}{x = t \supset t = x} \forall E \\
\frac{[x = t]^1}{t = x} \supset E \\
\frac{[A[t/x]]^2}{\text{subst}} \\
\frac{A}{x = t \supset A} \supset I^1 \\
\frac{x = t \supset A}{\forall x. x = t \supset A} \forall I \\
\frac{\forall x. x = t \supset A}{A[t/x] \supset \forall x. x = t \supset A} \supset I^2 \\
\frac{[\forall x. x = t \supset A]^1}{t = t \supset A[t/x]} \forall E \quad \frac{}{\forall x. x = x} \text{refl} \\
\frac{t = t \supset A[t/x]}{t = t} \forall E \\
\frac{t = t}{A[t/x]} \supset E \\
\frac{A[t/x]}{(\forall x. x = t \supset A) \supset A[t/x]} \supset I^1
\end{array}$$
$$\frac{\frac{\frac{[\forall x. \forall y. P]^1}{\forall y. P} \forall E}{P} \forall E}{\forall x. P} \forall I}{\forall y. \forall x. P} \forall I}{(\forall x. \forall y. P) \supset \forall y. \forall x. P} \supset I$$
$$\frac{\frac{\frac{[P]^3}{\exists x. P} \exists \text{I}}{[\exists y. P]^2} \frac{}{\exists y. \exists x. P} \exists \text{I}}{[\exists x. \exists y. P]^1} \frac{}{\exists y. \exists x. P} \exists \text{E}^3} \frac{}{\exists y. \exists x. P} \exists \text{E}^2} \frac{}{(\exists x. \exists y. P) \supset \exists y. \exists x. P} \supset \text{I}^1$$

(66) $\vdash \forall x. A \vee B = (\forall x. A) \vee B$ with $x \notin \text{FV}(B)$

$$\begin{array}{c}
 \frac{\frac{\frac{B \vee \neg B}{\text{lem}} \quad \frac{[B]^1}{(\forall x. A) \vee B} \vee I_2 \quad \frac{\frac{A}{\forall x. A} \vee I \quad \frac{(\forall x. A) \vee B}{\vee I_1}}{(\forall x. A) \vee B} \vee E^1}{(\forall x. A) \vee B} \supset I^2 \\
 \frac{\frac{[\forall x. A \vee B]^2}{A \vee B} \vee E \quad \frac{[A]^3}{A} \vee E^3 \quad \frac{[\neg B]^1 \quad [B]^3}{\perp} \neg E}{\frac{A \vee B}{\vee E^2} \quad \frac{[B]^2}{A \vee B} \vee I_2}{\frac{A \vee B}{\forall x. A \vee B} \vee I} \supset I^1 \\
 \hline
 \end{array}$$

(67) $\vdash (\forall x. A \supset B) \supset ((\exists x. A) \supset \exists x. B)$

$$\begin{array}{c}
 \frac{\frac{[\forall x. A \supset B]^2}{A \supset B} \vee E \quad \frac{[A]^3}{A} \supset E}{\frac{B}{\exists x. B} \exists I} \supset E \\
 \frac{[\exists x. A]^1}{\exists x. B} \exists E^3 \\
 \frac{\exists x. B}{(\exists x. A) \supset \exists x. B} \supset I^1 \\
 \hline
 \frac{(\forall x. A \supset B) \supset ((\exists x. A) \supset \exists x. B)}{\supset I^2}
 \end{array}$$

(68) $\vdash (\exists x. x = f(x) \wedge P[f(x)/x]) \supset \exists x. P$

$$\begin{array}{c}
 \frac{\frac{\frac{\forall x. \forall y. x = y \supset y = x}{\text{sym}} \quad \frac{\forall y. x = y \supset y = x}{\vee E} \quad \frac{x = f(x) \supset f(x) = x}{\vee E} \quad \frac{[x = f(x) \wedge P[f(x)/x]]^2}{x = f(x)} \wedge E_1}{\frac{f(x) = x}{P[f(x)/x]} \text{subst}} \wedge E_2 \\
 \frac{\frac{[x = f(x) \wedge P[f(x)/x]]^1}{\exists x. P} \exists E^2 \quad \frac{P}{\exists x. P} \exists I}{\frac{\exists x. P}{(\exists x. x = f(x) \wedge P[f(x)/x]) \supset \exists x. P} \supset I^1}
 \end{array}$$