MATHEMATICAL LOGIC — ASSIGNMENT TWO

(1) Prove $\vdash (A \lor \exists x.\, B) = (\exists x.\, (A \lor B))$ with $x \notin \mathsf{FV}(A)$.
Show a counterexample when $x \in \mathsf{FV}(A)$.

$$\dfrac{\dfrac{[\exists x.\, A \lor B]^1 \qquad \dfrac{[A \lor B]^2 \qquad \dfrac{[A]^3}{A \lor \exists x.\, B}\,{\scriptstyle \lor\mathrm{I}_1} \qquad \dfrac{\dfrac{[B]^3}{\exists x.\, B}\,{\scriptstyle \exists\mathrm{I}}}{A \lor \exists x.\, B}\,{\scriptstyle \lor\mathrm{I}_2}}{A \lor \exists x.\, B}\,{\scriptstyle \lor\mathrm{E}^3}}{A \lor \exists x.\, B}\,{\scriptstyle \exists\mathrm{E}^2}}{(\exists x.\, A \lor B) \supset A \lor \exists x.\, B}\,{\scriptstyle \supset\mathrm{I}^1}$$

$$\dfrac{\dfrac{[A \lor \exists x.\, B]^1 \qquad \dfrac{\dfrac{[A]^2}{A \lor B}\,{\scriptstyle \lor\mathrm{I}_1}}{\exists x.\, A \lor B}\,{\scriptstyle \exists\mathrm{I}} \qquad \dfrac{[\exists x.\, B]^2 \qquad \dfrac{\dfrac{[B]^3}{A \lor B}\,{\scriptstyle \lor\mathrm{I}_2}}{\exists x.\, A \lor B}\,{\scriptstyle \exists\mathrm{I}}}{\exists x.\, A \lor B}\,{\scriptstyle \exists\mathrm{E}^3}}{\exists x.\, A \lor B}\,{\scriptstyle \lor\mathrm{E}^2}}{A \lor (\exists x.\, B) \supset \exists x.\, A \lor B}\,{\scriptstyle \supset\mathrm{I}^1}$$

In arithmetic, let A be the formula "x is even", and let $B \equiv \bot$. Then $\exists x.\, A \lor B$ is true since $\exists x.\, A \lor B$ becomes $\exists x.\, A$ and there is an even number. Instead, if x is interpreted in 9, A is false, thus $A \lor \exists x.\, B$ becomes false.

(2) Show that a set Γ is maximal consistent if and only if it is consistent and for every formula A either $A \in \Gamma$ or $\neg A \in \Gamma$.

This is Proposition 10.4 in the slides.

(3) Show that every first-order theory T on the signature

$$\langle S; \{\circ \colon S \times S \to S\}; \{= \colon S \times S\}\rangle$$

having as models all the finite groups, has necessarily an infinite model.

Let $C = \{c_i \colon i \in \mathbb{N}\}$ be a set of constants.

Let $\Xi = \{c_i \neq c_j \colon i, j \in \mathbb{N} \land i \neq j\}$ be a set of axioms.

Consider a finite $F \subseteq T \cup \Xi$. Hence, there is finite number of axioms in F from Ξ, thus, in particular, there is finite number m of constants in C appearing in F.

Observe that there is necessarily a finite group $\mathcal{G}$ whose order (the number of its elements) is greater than m: for example, the permutation group on m, which has $m!$ elements. Hence, $\mathcal{G}$ is a model for F.

Thus, by the Compactness Theorem, $T \cup \Xi$ has a model $\mathcal{M}$, and all the c_i's are interpreted in distinct elements, thus $\mathcal{M}$ is infinite.

Hence, $\mathcal{M}$ is a model for T and it is infinite.

Date: April 26$^{\text{th}}$, 2023.