MATHEMATICAL LOGIC — ASSIGNMENT ONE

(1) Prove $\vdash (A \supset B) = (\neg B \supset \neg A)$.

$$
\begin{prooftree}
\AxiomC{$[A \supset B]^1$}
\AxiomC{$[A]^2$}
\RightLabel{$\supset$E}
\BinaryInfC{B}
\AxiomC{$[\neg B]^3$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\neg$I^2}
\UnaryInfC{$\neg A$}
\RightLabel{$\supset$I^3}
\UnaryInfC{$\neg B \supset \neg A$}
\RightLabel{$\supset$I^1}
\UnaryInfC{$(A \supset B) \supset (\neg B \supset \neg A)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\RightLabel{lem}
\UnaryInfC{$B \vee \neg B$}
\AxiomC{$[B]^1$}
\AxiomC{$[\neg B \supset \neg A]^2$}
\AxiomC{$[\neg B]^1$}
\RightLabel{$\supset$E}
\BinaryInfC{$\neg A$}
\AxiomC{$[A]^3$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\bot$E}
\UnaryInfC{B}
\RightLabel{$\vee$E^1}
\TrinaryInfC{B}
\RightLabel{$\supset$I^3}
\UnaryInfC{$A \supset B$}
\RightLabel{$\supset$I^2}
\UnaryInfC{$(\neg B \supset \neg A) \supset (A \supset B)$}
\end{prooftree}
$$

(2) Show that in a bounded lattice, every element is greater that $\bot$ and less than $\top$.

This is Proposition 5.5 in the slides.

(3) Prove that Pierce's Law $(((B \supset C) \supset B) \supset B)$ implies the Law of Excluded Middle $(A \vee \neg A)$.

Since Pierce's Law holds, we can assume an instance of it:

$$((A \vee \neg A \supset \bot) \supset A \vee \neg A) \supset A \vee \neg A \ .$$

Then

$$
\begin{prooftree}
\AxiomC{$((A \vee \neg A \supset \bot) \supset A \vee \neg A) \supset A \vee \neg A$}
\AxiomC{$[A \vee \neg A]^1$}
\AxiomC{$[A \vee \neg A \supset \bot]^2$}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\RightLabel{$\neg$I^1}
\UnaryInfC{$\neg(A \vee \neg A)$}
\AxiomC{$[A]^3$}
\RightLabel{$\vee$I$_1$}
\UnaryInfC{$A \vee \neg A$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\neg$I^3}
\UnaryInfC{$\neg A$}
\RightLabel{$\vee$I$_2$}
\UnaryInfC{$A \vee \neg A$}
\RightLabel{$\supset$I^2}
\UnaryInfC{$(A \vee \neg A \supset \bot) \supset A \vee \neg A$}
\RightLabel{$\supset$E}
\BinaryInfC{$A \vee \neg A$}
\end{prooftree}
$$

Date: March 29$^{\text{th}}$, 2023.