MATHEMATICAL LOGIC — ASSIGNMENT FOUR
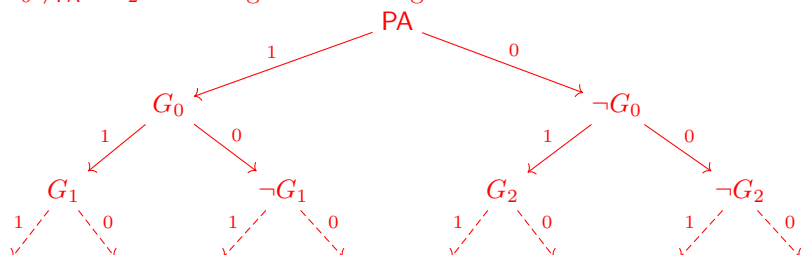
(1) Show a Heyting algebra in which $\neg\neg P \supset P$ is not valid.

Consider the Heyting algebra $0 < 1/2 < 1$. If P gets evaluated in $1/2$, $\neg P$ is evaluated in 0, so $\neg\neg P$ is evaluated in 1. Hence $\neg\neg P \supset P$ is evaluated in $1/2$ and not in 1, so it is not valid.

(2) Define when a relation is representable in Peano arithmetic.

See Definition 23.2.

(3) Show that Peano arithmetic has at least $2^{\aleph_0}$ non isomorphic models.

By the first incompleteness theorem there is a formula G_0 in Peano arithmetic PA such that $\nvdash_{\mathsf{PA}} G_0$ and $\nvdash_{\mathsf{PA}} \neg G_0$.

Then, by the same result there is a formula G_1 such that $G_0 \nvdash_{\mathsf{PA}} G_1$ and $G_0 \nvdash_{\mathsf{PA}} \neg G_1$. Also, there is a formula G_2 such that $\neg G_0 \nvdash_{\mathsf{PA}} G_2$ and $\neg G_0 \nvdash_{\mathsf{PA}} \neg G_2$. Iterating the reasoning ω times we obtain an infinite tree:

Each infinite branch is uniquely associated with a sequence of 0 and 1, meaning "right" and "left" respectively.

Each infinite branch has a model: indeed, let $\mathfrak{G}$ be the infinite collection of these formulæplus the axioms of PA; each finite set has a model as follows by the construction of the G's. Thus the whole branch has a model by the Compactness Theorem.

Counting these sequences means to know the number of models after ω steps. But each sequence s can be written as $0.s$, a real number in binary notation. The collection of all these real numbers is the unit interval $[0, 1]$ whose cardinality is $2^{\aleph_0}$.

Observe that two distinct sequences identify two non-isomorphic models since there is a point in the tree in which one model make true a sentence which the other model make false, and isomorphic models make true the very same set of sentences.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.